Tutorial

Geometric Computing in Computer Graphics using Conformal Geometric Algebra

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Abstract

*Early in the development of Computer Graphics it was realized that projective geometry was well suited for the representation of transformations. Now, it seems that another change of paradigm is lying ahead of us based on **Geometric Computing using Conformal Geometric Algebra**.*

Due to its geometric intuitiveness, elegance and simplicity, the underlying Conformal Geometric Algebra appears to be a promising mathematical tool for Computer Graphics and Animations.

In this tutorial paper we introduce into the basics of the Conformal Geometric Algebra and show its advantages based on two Computer Graphics applications.

*First, we will present an algorithm for the **Inverse Kinematics** of a robot that you are able to comprehend without prior knowledge of Geometric Algebra. We expect that here you will obtain the basic knowledge for developing your own algorithm afterwards.*

*Second, we will show how easy it is in Conformal Geometric Algebra, to **fit the best suitable object in a set of points** , whether it is a plane or a sphere.*

1. Introduction

In this tutorial paper, we focus on the introduction of the 5D **Conformal Geometric Algebra** which is an extension of the 4D Projective Geometric Algebra.

While points and vectors are normally used as basic geometric entities, in Conformal Geometric Algebra we have a wider variety of basic objects. For example, spheres and
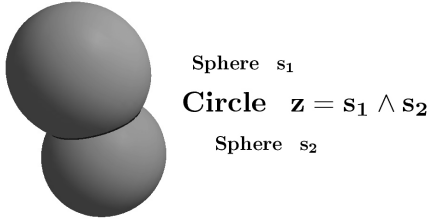

Figure 1: *Intersection of two spheres*

circles are simply represented by algebraic objects. To represent a circle you only have to intersect two spheres, which can be done with a basic algebraic operation. Alternatively you can simply combine three points to obtain the circle through these three points.

Besides the construction of algebraic entities, kinematics can also be expressed in Geometric Algebra. We present an algorithm for the inverse kinematics of a robot. The geometrically intuitive operations of Geometric Algebra make it easy to compute the joint angles of a robot which need to be set in order for the robot to reach its new position.

Spheres, points and planes are all represented as vectors in Conformal Geometric Algebra. We will see, that the inner product of these objects is a good distance measure between them.

Based on these observations we will see in our second application how easy it is, to fit the best suitable object in a set of points, whether it is a plane or a sphere.

Some more detailed introductions to the Conformal Geometric Algebra will be found in [7] and [10], some Geometric Algebra tutorials will be found in [2], [3], [8], [12], [14], [16] and some applications in [4], [5] and [15].

2. The products of the Conformal Geometric Algebra

The main products of the Geometric Algebra are the geometric product, the inner product and the outer product. In this paper we focus on the **outer product** (indicated by '∧') and the **inner product** (indicated by '·'). We will use the outer product mainly for the construction and intersection of geometric objects while the inner product will be used for the computation of angles and distances.

Table 1: *list of the conformal geometric entities*

entity	representation 1	representation 2
Point	$P = \mathbf{x} + \frac{1}{2}\mathbf{x}^2 e_\infty + e_0$	
Sphere	$s = P - \frac{1}{2}r^2 e_\infty$	$s^* = x_1 \wedge x_2 \wedge x_3 \wedge x_4$
Plane	$\pi = \mathbf{n} + d e_\infty$	$\pi^* = x_1 \wedge x_2 \wedge x_3 \wedge e_\infty$
Circle	$z = s_1 \wedge s_2$	$z^* = x_1 \wedge x_2 \wedge x_3$
Line	$l = \pi_1 \wedge \pi_2$	$l^* = x_1 \wedge x_2 \wedge e_\infty$
Point Pair	$Pp = s_1 \wedge s_2 \wedge s_3$	$Pp^* = x_1 \wedge x_2$

3. The outer product and the basic geometric entities in Conformal Geometric Algebra

Table 1 lists the two representations of the geometric entities in Conformal Geometric Algebra. Please find details in [8] and [11].

In this table $\mathbf{x}$ and $\mathbf{n}$ are marked bold to indicate that they represent 3D entities as linear combination of the 3D base vectors e_1, e_2 and e_3.

$$\mathbf{x} = x_1 e_1 + x_2 e_2 + x_3 e_3 \qquad (1)$$

The additional two base vectors are indicated by

- e_0 representing the 3D origin
- e_∞ representing the point at infinity

The $\{s_i\}$ represent different spheres and the $\{\pi_i\}$ different planes.

The two representations are **dual** to each other. In order to switch between the two representations you can use the dual operator which is indicated by '*'. Depending on the application and convenience, one of these two sets of representations is selected as standard representation. We use representation 1 as standard representation and representation 2 as dual representation.

In representation 2 the outer product '∧' indicates the construction of geometric objects with the help of points x_i that lie on it. E. g. a sphere is defined by 4 points ($x_1 \wedge x_2 \wedge x_3 \wedge x_4$) determining the sphere.

In representation 1 the meaning of the outer product is the intersection of geometric entities. E. g. a circle is defined by the intersection of two spheres ($s_1 \wedge s_2$).

Please refer to figure 1.

3.1. Points

In order to represent points in 5D conformal space, the original 3D point $\mathbf{x}$ is extended to a 5D vector according to the equation

$$P = \mathbf{x} + \frac{1}{2}\mathbf{x}^2 e_\infty + e_0 \qquad (2)$$

where $\mathbf{x}^2$ is the well-known scalar product

$$\mathbf{x}^2 = x_1^2 + x_2^2 + x_3^2 \tag{3}$$

E. g. for the 3D origin (0,0,0) we get

$$P(0,0,0) = e_0 \tag{4}$$

or for the 3D point (0,1,0)

$$p_y = P(0,1,0) = e_2 + \frac{1}{2}e_\infty + e_0 \tag{5}$$

3.2. Spheres

A sphere is represented with the help of its center point P and its radius r.

$$s = P - \frac{1}{2}r^2 e_\infty \tag{6}$$

Note that the representation of a point is simply a sphere with radius zero.

A sphere can also be represented with the help of 4 points that lie on it.

$$s^* = x_1 \wedge x_2 \wedge x_3 \wedge x_4 \tag{7}$$

3.3. Planes

A plane is defined by

$$\pi = \mathbf{n} + d e_\infty \tag{8}$$

$\mathbf{n}$ refers to the 3D normal vector of the plane π and d is the distance to the origin.

A plane can also be defined with the help of 3 points that lie on it and the point at infinity.

$$\pi^* = x_1 \wedge x_2 \wedge x_3 \wedge e_\infty \tag{9}$$

Notice that a plane is a sphere with infinite radius.

3.4. Circles

A circle is defined by the intersection of two spheres

$$z = s_1 \wedge s_2 \tag{10}$$

or with the help of three points that lie on it

$$z^* = x_1 \wedge x_2 \wedge x_3 \tag{11}$$

3.5. Lines

A line is defined by the intersection of two planes

$$l = \pi_1 \wedge \pi_2 \tag{12}$$

or with the help of two points that lie on it and the point at infinity

$$l^* = x_1 \wedge x_2 \wedge e_\infty \tag{13}$$

E. g. the y-axis l_y can be described by

$$l_y^* = e_0 \wedge p_y \wedge e_\infty \tag{14}$$

where e_0 represents the origin (see equation (4)) and p_y is the point of equation (5).

Also notice that a line can be regarded as a circle with infinite radius.

3.6. Point Pairs

A point pair is defined by the intersection of three spheres

$$Pp = s_1 \wedge s_2 \wedge s_3 \tag{15}$$

or with the help of the two points

$$Pp^* = x_1 \wedge x_2 \tag{16}$$

4. The Inner Product and Angles

The inner product of 3D vectors corresponds to the well-known scalar product. The 3D base vectors e_1, e_2, e_3 square to 1

$$e_1^2 = e_2^2 = e_3^2 = 1 \tag{17}$$

For instance, the length of the normal vector of equation (8)

$$\mathbf{n} = n_1 e_1 + n_2 e_2 + n_3 e_3 \tag{18}$$

results in

$$|\mathbf{n}| = \sqrt{n_1^2 + n_2^2 + n_3^2} = 1 \tag{19}$$

Because of the specific metric of the conformal space, the additional base vectors e_o^2, e_∞^2 square to 0

$$e_o^2 = e_\infty^2 = 0 \tag{20}$$

and their inner product results in

$$e_\infty \cdot e_o = -1 \tag{21}$$

Angles between two objects o_1, o_2 like two lines or two planes can be computed using the inner product of the normalized dual objects.

$$cos(\theta) = \frac{o_1^* \cdot o_2^*}{|o_1^*| \, |o_2^*|} \tag{22}$$

or

$$angle(o_1^*, o_2^*) = \arccos \frac{o_1^* \cdot o_2^*}{|o_1^*| \, |o_2^*|} \tag{23}$$

Please refer to [8] for more details.
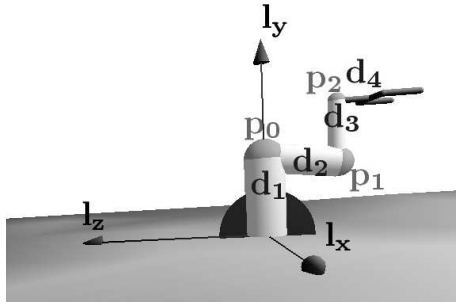
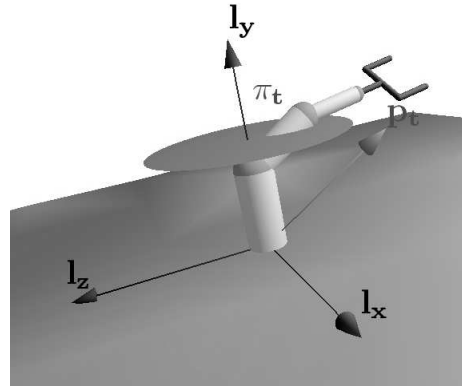**Figure 2:** *kinematic chain of a robot*

Figure 3: *InverseKinematics.clu*

5. Application 1 : Inverse Kinematics

Objects like robots or virtual humans can be modelled as a set of rigid links connected together at various joints. These objects are described as kinematic chains.

The simple robot presented in figure 2 consists of 3 links and one gripper

- the 3 joint points are called p_0, p_1 and p_2
- the 3 link distances are called d_1, d_2 and d_3
- the distance from the last joint p_2 to the 'T' intersection of the gripper is called d_4

It has 5 degrees of freedom (DOF) by means of the following 5 joint angles $\theta_1 .. \theta_5$:

- θ_1 : rotate robot (around l_y)
- $\theta_2, \theta_3, \theta_4$: acting in plane π_1
- θ_5 : rotate gripper

where the plane π_1 is defined by the origin e_0, the point p_y on the y-axis and the target point p_t. According to equation (9) we get

$$\pi_1^* = e_0 \wedge p_y \wedge p_t \wedge e_\infty \qquad (24)$$

This section is concerned with the inverse problem of finding the joint angles in terms of a target position p_t and an orientation of the gripper plane π_t.

In Conformal Geometric Algebra, this so-called **inverse kinematics** can be done in a geometrically very intuitive way due to its easy handling of intersections of spheres, circles, planes etc.

Our approach is based on the papers [1] and [9]. For ease of use we define the gripper plane π_t as parallel to the ground plane. Since a plane can be described using equation (8) we get

$$\pi_t = e_2 + p_{t,y} \, e_\infty \qquad (25)$$

where $p_{t,y}$ is the y-coordinate of the target point p_t.

In the following steps we will first calculate the 3 locations p_0, p_1, p_2. Based on these points we will be able to calculate the 5 joint angles $\theta_1 .. \theta_5$.
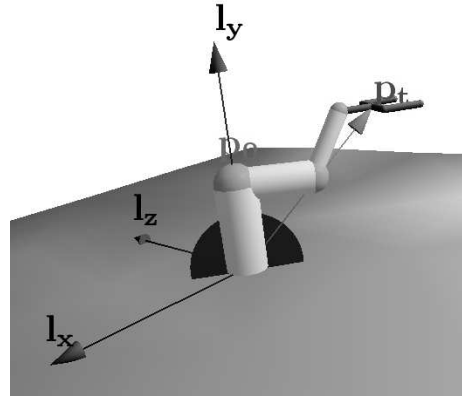
5.1. Step 1

Figure 4: *InverseKinematics.clu, Step 1*

In the first step point p_0 is calculated. Its 3D representation is $(0, d_1, 0)$.
Using equation (2) we get

$$p_0 = d_1 e_2 + \frac{1}{2} d_1^2 e_\infty + e_0 \qquad (26)$$

5.2. Step 2

In the second step point p_2 is calculated.
p_2 is the joint location of the last link of the robot. This means that it has to lie on the sphere S_t with the center point p_t and with the length d_4 of the displacement between p_t and p_2 as radius. Using equation (6) we get

$$S_t = p_t - \frac{1}{2} d_4^2 e_\infty \qquad (27)$$

Since the gripper also has to lie in the orientation plane π_t, we have to intersect it with S_t. The result is the circle z_t (see
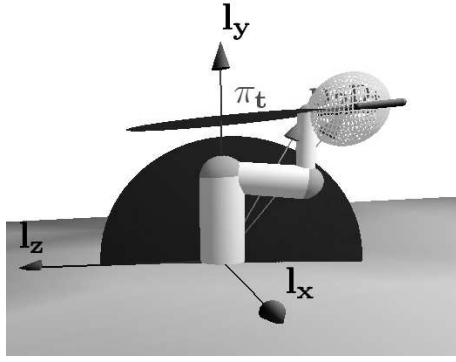
Figure 5: *InverseKinematics.clu, Step 2*

equation (10)).

$$z_t = S_t \wedge \pi_t \tag{28}$$

Please notice that a plane is a sphere with infinite radius.

Since p_2 also has to lie in the plane π_1, its intersection with the circle z_t results in a point pair. (see equation (15))

$$Pp_2 = z_t \wedge \pi_1 \tag{29}$$

From the mechanics point of view, only one of these two points is applicable which we choose as our point p_2. Please find further details on dissecting a point pair in [8].

5.3. Step 3
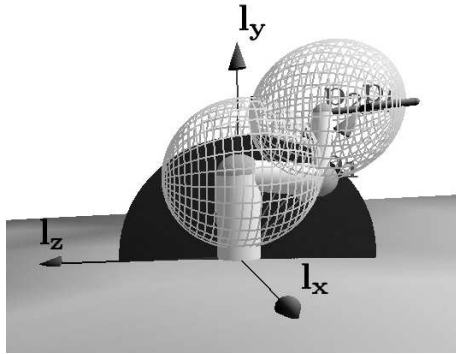
Figure 6: *InverseKinematics.clu, Step 3*

In the third step point p_1 is calculated.

Computing this point is usually a difficult task because it is the intersection of two circles. However, using Conformal Geometric Algebra we can determine it by intersecting the spheres S_1 and S_2 with the plane π_1.
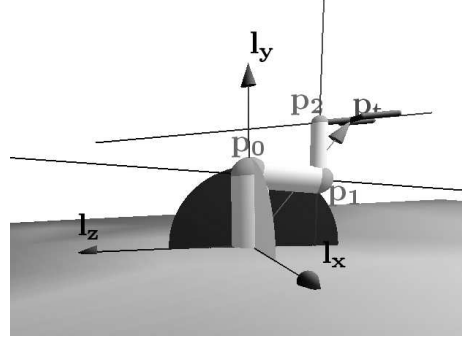
$$S_1 = p_0 - \frac{1}{2}d_2^2 e_\infty \tag{30}$$

$$S_2 = p_2 - \frac{1}{2}d_3^2 e_\infty \tag{31}$$

$$Pp_1 = S_1 \wedge S_2 \wedge \pi_1 \tag{32}$$

Again, we have to choose one point from the resulting point pair.

5.4. Step 4 : Computation of the joint angles

Figure 7: *InverseKinematics.clu, Step 4*

First, all the auxiliary planes and lines, that are needed for the computation of the angles of the joints are calculated. We need

- the plane π_2 spanned by the x-axis and the y-axis. Since the z-axis is perpendicular to this plane, we get

$$\pi_2 = e_3 \tag{33}$$

- the line l_1 through p_0 and p_1

$$l_1^* = p_0 \wedge p_1 \wedge e_\infty \tag{34}$$

- the line l_2 through p_1 and p_2

$$l_2^* = p_1 \wedge p_2 \wedge e_\infty \tag{35}$$

- the line l_3 through p_2 and p_t

$$l_3^* = p_2 \wedge p_t \wedge e_\infty \tag{36}$$

Now, we are able to compute all the joint angles

$$\theta_1 = \angle(\pi_1, \pi_2) \tag{37}$$

$$\theta_2 = \angle(l_1, l_y) \tag{38}$$

$$\theta_3 = \angle(l_1, l_2) \tag{39}$$

$$\theta_4 = \angle(l_2, l_3) \tag{40}$$

using the equation (23) with o_1, o_2 being either two lines or two planes.
In our simplified example

$$\theta_5 = 0 \tag{41}$$

since the gripper should be parallel to the ground plane.

6. The Inner Product and Distances

In the Conformal Geometric Algebra points, planes and spheres are represented as vectors.
The inner product of these objects results in a scalar and can be used as a measure for distances.
In this section, we will see, that the inner product $P \cdot S$ of two vectors P and S can be used for tasks like

- the Euclidean distance between two points
- the distance between one point and one plane
- the decision whether a point is inside or outside of a sphere

6.1. Vectors in Conformal Geometric Algebra

A vector in Conformal Geometric Algebra can be written as

$$S = \acute{s_1}e_1 + \acute{s_2}e_2 + \acute{s_3}e_3 + \acute{s_4}e_\infty + \acute{s_5}e_0 \qquad (42)$$

The meaning of the two additional coordinates e_0 and e_∞ is as follows :

	$\acute{s_5} = 0$	$\acute{s_5} \neq 0$
$\acute{s_4} = 0$	plane through origin	sphere/point through origin
$\acute{s_4} \neq 0$	plane	sphere/point

The multiplication with a constant $k \neq 0$ leads always to the same geometric object.
Division by $\acute{s_5} \neq 0$ leads to

$$S = s_1e_1 + s_2e_2 + s_3e_3 + s_4e_\infty + e_0 \qquad (43)$$

representing a sphere S with center point $\mathbf{s}$ and radius r

$$S = \mathbf{s} + s_4e_\infty + e_0 \qquad (44)$$

with

$$s_4 = \frac{1}{2}(\mathbf{s}^2 - r^2) = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2 - r^2)$$

Points are degenerate spheres with radius $r = 0$.

$$P = \mathbf{s} + \frac{1}{2}\mathbf{s}^2 e_\infty + e_0 \qquad (45)$$

Note : when inserting this formula in equation (6) the result corresponds to the equation (44).
Planes are degenerate spheres with infinite radius. They are represented as a vector with $s_5 = 0$

$$S = s_1e_1 + s_2e_2 + s_3e_3 + s_4e_\infty \qquad (46)$$

This corresponds to the equation (8) if we transform it to an expression with normal vector by dividing with

$$|\mathbf{s}| = \sqrt{s_1^2 + s_2^2 + s_3^2}$$

.

6.2. The Inner Product of Vectors

The inner product between a vector P and a vector S is defined by

$$P \cdot S = (\mathbf{p} + p_4\mathbf{e}_\infty + p_5\mathbf{e}_o) \cdot (\mathbf{s} + s_4\mathbf{e}_\infty + s_5\mathbf{e}_o)$$

Let us now translate the inner product to an expression in Euclidean space

$$P \cdot S = \mathbf{p} \cdot \mathbf{s} + s_4 \underbrace{\mathbf{p} \cdot \mathbf{e}_\infty}_{0} + s_5 \underbrace{\mathbf{p} \cdot \mathbf{e}_o}_{0}$$

$$+ p_4 \underbrace{\mathbf{e}_\infty \cdot \mathbf{s}}_{0} + p_4 s_4 \underbrace{\mathbf{e}_\infty^2}_{0} + p_4 s_5 \underbrace{\mathbf{e}_\infty \cdot \mathbf{e}_o}_{-1}$$

$$+ p_5 \underbrace{\mathbf{e}_o \cdot \mathbf{s}}_{0} + p_5 s_4 \underbrace{\mathbf{e}_o \cdot \mathbf{e}_\infty}_{-1} + p_5 s_5 \underbrace{\mathbf{e}_o^2}_{0}$$

Based on the equations of section 4 and the fact that a 3D vector is perpendicular to the additional base vectors this results in

$$P \cdot S = \mathbf{p} \cdot \mathbf{s} - p_5 s_4 - p_4 s_5 \qquad (47)$$

or

$$P \cdot S = p_1 s_1 + p_2 s_2 + p_3 s_3 - p_5 s_4 - p_4 s_5$$

6.3. Distances between points

In the case of P and S being points we get

$$p_4 = \frac{1}{2}\mathbf{p}^2, \quad p_5 = 1$$

$$s_4 = \frac{1}{2}\mathbf{s}^2, \quad s_5 = 1$$

The inner product of these points is according to equation (47)

$$P \cdot S = \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}\mathbf{s}^2 - \frac{1}{2}\mathbf{p}^2$$

$$= p_1 s_1 + p_2 s_2 + p_3 s_3 - \frac{1}{2}(s_1^2 + s_2^2 + s_3^2) - \frac{1}{2}(p_1^2 + p_2^2 + p_3^2)$$

$$= -\frac{1}{2}(s_1^2 + s_2^2 + s_3^2 + p_1^2 + p_2^2 + p_3^2 - 2p_1 s_1 - 2p_2 s_2 - 2p_3 s_3)$$

$$= -\frac{1}{2}((s_1 - p_1)^2 + (s_2 - p_2)^2 + (s_3 - p_3)^2)$$

$$= -\frac{1}{2}(\mathbf{s} - \mathbf{p})^2$$

We recognize that the square of the Euclidean distance of the inhomogenous points corresponds to the inner product of the homogenous points multiplied by -2.

$$(\mathbf{s} - \mathbf{p})^2 = -2(P \cdot S) \qquad (48)$$

6.4. Distance between points and planes

For a vector P representing a point we get

$$p_4 = \frac{1}{2}\mathbf{p}^2, \quad p_5 = 1$$

For a vector S representing a plane with normal vector $\mathbf{n}$ and distance d we get

$$\mathbf{s} = \mathbf{n}, \quad s_4 = d, \quad s_5 = 0$$

The inner product of point and plane is according to equation (47)

$$P \cdot S = \mathbf{p} \cdot \mathbf{n} - d \tag{49}$$

representing the Euclidean distance of a point and a plane.

6.5. is a point inside or outside of a sphere ?

We will see now that the inner product of a point and a sphere can be used for the decision of whether a point is inside or outside of a sphere.

For a vector P representing a point we get

$$p_4 = \frac{1}{2}\mathbf{p}^2, \quad p_5 = 1$$

For a vector S representing a sphere we get

$$s_4 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2 - r^2), \qquad s_5 = 1$$

The inner product of point and sphere is according to equation (47)

$$P \cdot S = \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}(s^2 - r^2) - \frac{1}{2}\mathbf{p}^2$$

$$= \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}\mathbf{s}^2 + \frac{1}{2}r^2 - \frac{1}{2}\mathbf{p}^2$$

$$= \frac{1}{2}r^2 - \frac{1}{2}(\mathbf{s}^2 - 2\mathbf{p} \cdot \mathbf{s} - \mathbf{p}^2)$$

$$= \frac{1}{2}r^2 - \frac{1}{2}(\mathbf{s} - \mathbf{p})^2$$

We get

$$2(P \cdot S) = r^2 - (\mathbf{s} - \mathbf{p})^2 \tag{50}$$

That is equal to the square of the radius minus the square of the distance between the point and the center point of the sphere.
Based on this observation we can see that

$P \cdot S > 0 : \mathbf{p}$ is inside of the sphere
$P \cdot S = 0 : \mathbf{p}$ is on the sphere
$P \cdot S < 0 : \mathbf{p}$ is outside of the sphere

7. Application 2 : Fitting Planes or Spheres into Point Sets

In this section, a point set $P = \mathbf{p_i}$, $i \in \{1,...,n\}$, $\mathbf{p_i} \in \mathbb{R}^3$ is approximated with the help of the best fitting plane or sphere.

Plane and sphere in conformal space are vectors of the form

$$S = s_1e_1 + s_2e_2 + s_3e_3 + s_4e_\infty + s_5e_0 \tag{51}$$

while the points $\mathbf{p_i}$ are specific vectors of the form

$$X_i = \mathbf{p_i} + 0.5\mathbf{p_i}^2 e_\infty + e_0 \tag{52}$$

7.1. Approach

In order to solve the fitting problem we

- use the distance measure of the previous section between point and sphere/plane with the help of the inner product.
- make a least squares approach to minimize the squares of the distances between the points and the sphere/plane.
- solve the resulting eigenvalue problem.

7.2. Distance Measure

A distance measure between a point X_i and the sphere/plane S can be defined in Conformal Geometric Algebra with the help of their inner product

$$X_i \cdot S = (\mathbf{p_i} + 0.5\mathbf{p_i}^2 e_\infty + e_0) \cdot (\mathbf{s} + s_4e_\infty + s_5e_0) \tag{53}$$

According to equation (47) this results in

$$X_i \cdot S = \mathbf{p_i} \cdot \mathbf{s} - s_4 - \frac{1}{2}s_5\mathbf{p_i}^2$$

or

$$X_i \cdot S = \sum_{j=1}^{5} w_{i,j}s_j \tag{54}$$

with

$$w_{i,k} = \begin{cases} p_{i,k} & : & k \in \{1,2,3\} \\ -1 & : & k = 4 \\ -\frac{1}{2}\mathbf{p_i}^2 & : & k = 5 \end{cases}$$

7.3. Least Squares Approach

In the least-squares sense we consider the minimum of the squares of the distances between all the points and the plane/sphere

$$min \sum_{i=1}^{n} (X_i \cdot S)^2 \tag{55}$$

In order to obtain the minimum this can be rewritten in bilinear form to

$$min(s^T B s) \tag{56}$$

with

$$s^T = (s_1, s_2, s_3, s_4, s_5)$$

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & b_{1,5} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,5} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & b_{4,5} \\ b_{5,1} & b_{5,2} & b_{5,3} & b_{5,4} & b_{5,5} \end{pmatrix}$$

$$b_{j,k} = \sum_{i=1}^{n} w_{i,j} w_{i,k}$$

The matrix B is symmetric since $b_{j,k} = b_{k,j}$. Without loss of generality we consider only normalized results $s^T s = 1$. A conventional approach to such a constrained optimization problem is to introduce the Lagrangian

$$L = s^T B s - \lambda s^T s,$$

$$s^T s = 1,$$

$$B^T = B$$

Necessary conditions for a minimum are

$$0 = \nabla L = 2 \cdot (Bs - \lambda s) = 0$$

$$\rightarrow Bs = \lambda s$$

The solution of the minimization problem is given as the Eigenvector of B that corresponds to the smallest Eigenvalue. (see [6] for details)

7.4. Example

Let us have a look on an example with 5 points.

Point	x	y	z
p_1	1	0	0
p_2	1	1	0
p_3	0	0	1
p_4	0	1	1
p_5	-1	0	1

The result of the least squares calculation results in

$$S = -0.301511e_1 + 0.301511e_2 - 0.301511e_3$$

$$-0.603023e_\infty + 0.603023e_0$$

Another representation of this object is

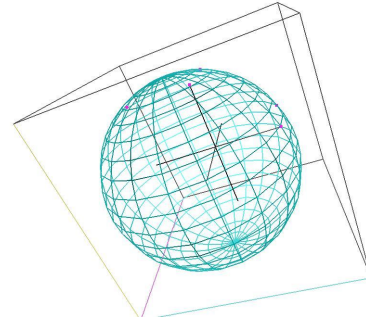$$S = -0.5e_1 + 0.5e_2 - 0.5e_3 - e_\infty + e_0$$

Figure 8: *Fit of a sphere.*

This corresponds to a sphere with the center point $\mathbf{s} = (-0.5, 0.5, -0.5)$ and the square of the radius as $r^2 = 2.75$. (see figure 8)

Now, let us change the fifth point in order that all the points are within one plane.

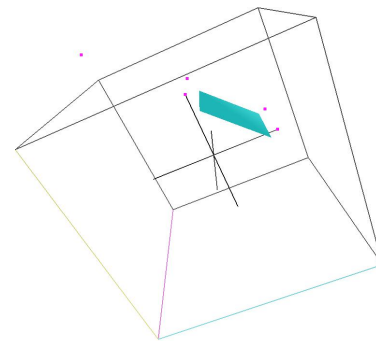| Point	x	y	z
p_1	1	0	0
p_2	1	1	0
p_3	0	0	1
p_4	0	1	1
p_5	-1	0	2

Figure 9: *Fit of a plane.*

Now, the result is

$$S = 0.57735 * e1 + 0.57735 * e3 + 0.57735 * e_\infty$$

representing a plane. (see figure 9)

8. Visual development of algorithms

We use the OpenSource **CLUCalc** software to **calculate with Geometric Algebra** and to **visualize the results** of these calculations. CLUCalc is freely available for download at [13]. With the help of the CLUCalc Software you are able to edit and run Scripts called **CLUScripts**. A screenshot of CLUCalc can be seen in figure 10.
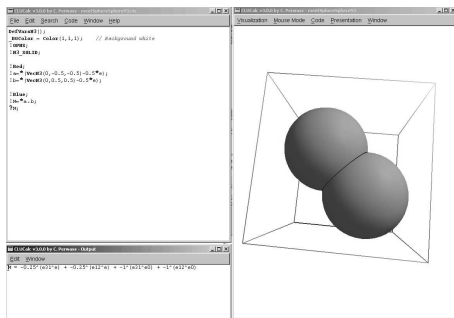

Figure 10: *Screenshot of the CLUCalc windows*

CLUCalc provides the following three windows

- editor window
- visualization window
- output window

The Inverse Kinematics algorithm, described in this paper, is implemented using CLUCalc. The corresponding CLUScripts (InverseKinematics.clu is the main file) can be downloaded from the homepage

```
http://www.gris.informatik.tu-
darmstadt.de/~dhilden/
```

There is almost a one to one correspondence between formulae and codes. E. g. the computations of step 3 are easily done as follows

```
S1 = p0 - 0.5*d2*d2*einf;
S2 = p2 - 0.5*d3*d3*einf;
Pp1 = S1^S2^PI1;
// choose one of the two points
p1 = DissectSecond(*Pp1);
```

For details regarding CLUScript please refer to the CLUCalc online help [13].
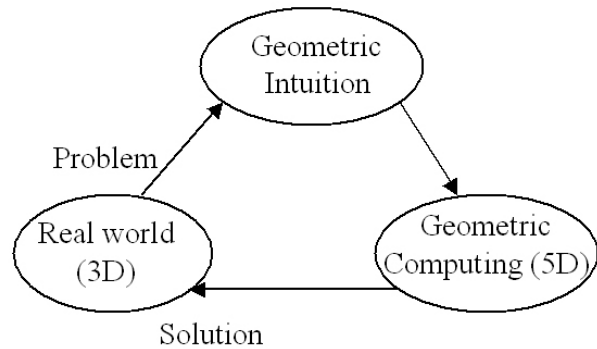
9. Conclusion

In this paper, we solved our application problems using Geometric Computing. Therefore we transferred our 3D problem with the help of geometric intuition into the 5D Conformal Geometric Algebra and got our results back into the real 3D world.

Geometric Computing is applicable in many different engineering scenarios and provides a straightforward and intuitive problem solving approach.

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