Geometric Algebra Approach to Fluid Dynamics

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Abstract In this work we will use geometric algebra to prove a number of well known theorems central to the field of fluid dynamics, such as Kelvin's Circulation Theorem and Helmholtz' Theorem, showing that it is accessible by geometric algebra methods and that these methods facilitate the representation of and calculation with fluid dynamics concepts. Then we propose a generalization of the stream function to arbitrary dimensions, extending its explanatory power to higher-dimensional flows. We will show how this extended stream function behaves on streamlines and we will relate it to the curl of the flow's vector field, i.e. its vorticity.

1 Introduction

While geometric algebra methods have been successfully applied to a variety of problems in the recent past, including computer vision [10], robotics [3] [8] and relativistic electro-magnetic field theory [1] [5], its applications to the field of fluid dynamics have been restricted to isolated subjects or specialized problem descriptions, e.g. [4] [6] [9]. The systematic development of a comprehensive fluid dynamics calculus using geometric algebra methods is still an open task.

In this work we will use geometric algebra to prove a number of well known theorems central to the field of fluid dynamics, showing that some conclusions follow neatly from geometric properties (e.g. of vector fields) encoded in the calculational rules of the geometric product. Then we propose a generalization of the *stream*

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function to arbitrary dimensions. Classically, the stream function is defined for two-dimensional flows, where it can be used to calculate stream lines or the (complex) potential of a flow. Generalizing the stream function to arbitrary dimensions offers extended possibilities of analyzing a fluid flow.

We will proceed as follows. In section 2 we will give a short introduction into what we consider advanced concepts of geometric algebra, like vector differentiation and integration. Section 3 will introduce the reader to some pecularities of fluid dynamics on a rather algebraic level, to an extent that is needed to understand this work. For a more extended and figurative introduction, see [2]. In section 4 we use geometric algebra to prove a number of theorems central to fluid dynamics, such as Kelvin's Circulation Theorem and Helmholtz' Theorem. Also, we will show how to define the stream function, which is recognized as useful for the description of 2D flows, in arbitrary dimensions. Finally, in section 5, we will summarize our findings.

2 Geometric Algebra

We assume that the reader is familiar with the basics of geometric algebra, i.e. the inner, outer and geometric product, the way these apply to vectors, how they extend to higher grade objects, the role of the pseudoscalar etc.

In the following we will give a brief introduction into vector differentiation and directed integration. For a comprehensive and more axiomatic introduction we refer the reader to [7]. [5] gives an introduction that might be considered more accessible.

2.1 Vector Differentiation

The vector derivative ∇ unites the algebraic properties of a vector with those of an operator. This becomes clear, if one expands it in terms of a basis. Given a coordinate frame $\{\mathbf{e}_k\}$ and its reciprocal frame $\{\mathbf{e}^k\}$, one can write

$$\nabla = \sum_k \mathbf{e}^k \mathbf{e}_k \cdot \nabla = \sum_k \mathbf{e}^k \frac{\partial}{\partial x^k}\,, \tag{1}$$

which shows that ∇ inherits the properties of the partial derivatives – i.e. chain rule and product rule apply – while at the same time it plays the role of a vector in any product. The application of the vector derivative to scalar and vector-valued functions or fields, respectively, may serve as an illustration of its calculational power.

Let $p(\mathbf{x})$ be a scalar-valued field, i.e. a function depending on n-dimensional position $\mathbf{x}$. Then

$$\nabla p(\mathbf{x}) = \sum_k \mathbf{e}^k \frac{\partial p(\mathbf{x})}{\partial x^k}\,, \tag{2}$$

is the vector-valued gradient of the scalar field $p(\mathbf{x})$.

Things get more interesting when we apply ∇ to a vector-valued function, e.g. the vector field $\mathbf{u}(\mathbf{x})$. The full vector derivative yields

$$\nabla \mathbf{u}(\mathbf{x}) = \nabla \cdot \mathbf{u}(\mathbf{x}) + \nabla \wedge \mathbf{u}(\mathbf{x}), \tag{3}$$

a general multivector consisting of two terms of grade zero and two, respectively. The scalar term $\nabla \cdot \mathbf{u}(\mathbf{x})$, which is the *divergence* of $\mathbf{u}(\mathbf{x})$ and a pure bivector $\nabla \wedge \mathbf{u}(\mathbf{x})$, which is the vector field's *curl*. Note that neither contains an abuse of notation, but both seamlessly integrate into the geometric algebra calculus.

Moreover, it is possible to extend the vector derivative to general multivectors and take the derivative $\nabla F(\mathbf{x})$ of any multivector-valued function $F(\mathbf{x})$ with respect to a vector argument.

As a *vector* the position of ∇ in any geometric algebra expression is not arbitrary, since the geometric product is not commutative. As a consequence it no longer suffices to determine the target of the *operator* ∇. The following rule of notation may help to disambiguate things.

- In the absence of brackets, ∇ acts on the quantity to its immediate right.
- When ∇ is followed by brackets, it acts on all of the terms inside the brackets.
- When ∇ acts on a multivector, which is not on its right, the scope will be signified by overdots.
- If ∇ has a subscript, then that indicates the variable with respect to which the derivative has to be taken.

Consider the derivative of a nested function. Let $\mathbf{u}(\mathbf{v}(\mathbf{x}))$ be such a function, with $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{x}$ vectors. Then, by the chain rule

$$\begin{aligned} \nabla_{\mathbf{x}} \mathbf{u} &= \sum \frac{\partial \mathbf{u}}{\partial x^i} \mathbf{e}^i \\ &= \sum \frac{\partial \mathbf{u}}{\partial v^j} \frac{\partial v^j}{\partial x^i} \mathbf{e}^i \\ &= \dot{\nabla}_{\mathbf{x}} (\dot{\mathbf{v}}(\mathbf{x}) \cdot \acute{\nabla}_{\mathbf{v}}) \acute{\mathbf{u}}(\mathbf{v}(\mathbf{x})) . \end{aligned} \tag{4}$$

With that it is possible to define the directional derivative of any multivector field in the direction of a vector $\mathbf{a}$ in terms of the geometric product as

$$(\mathbf{a} \cdot \nabla) F(\mathbf{x}) = \frac{1}{2} \left(\mathbf{a} \nabla F(\mathbf{x}) + \dot{\nabla} \mathbf{a} \dot{F}(\mathbf{x}) \right) . \tag{5}$$

2.2 Directed Integration

Consider a curve C in n-dimensional space. Furthermore, assume that C has the parametric representation $\mathbf{x}(s)$ with endpoints $\mathbf{a} = \mathbf{x}(\alpha)$ and $\mathbf{b} = \mathbf{x}(\beta)$. Even though there exist multiple alternative definitions, for the purpose of this work let us define the *directed line integral* in terms of the scalar integral by

$$\int_C F(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\alpha^\beta F(\mathbf{x})\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\mathrm{d}s\,. \tag{6}$$

The term

$$\mathrm{d}\mathbf{x} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\mathrm{d}s = \mathbf{e}(\mathbf{x})\mathrm{d}s \tag{7}$$

is called *directed line element*. The argument $\mathbf{x}$ of $\mathbf{e}$ often is suppressed.

Three things are notable about this definition. First, the vector $\mathbf{e}(\mathbf{x})$ is the tangent vector of the curve C in point $\mathbf{x}$, spanning the *tangent space* there. Secondly, the measure $\mathrm{d}\mathbf{x}$ preserves a sense of direction, since it is defined as a vector-valued infinitesimal quantity. And finally, the product between the integrand and the measure is to be read as a *geometric product*. That is, for example, for any vector field $\mathbf{u}(\mathbf{x})$ the integral can be split into

$$\int_C \mathbf{u}(\mathbf{x})\mathrm{d}\mathbf{x} = \int_C \mathbf{u}(\mathbf{x})\cdot\mathrm{d}\mathbf{x} + \int_C \mathbf{u}(\mathbf{x})\wedge\mathrm{d}\mathbf{x}\,. \tag{8}$$

In analogy to (7) one determines the directed line elements for coordinate curves in higher dimensions by

$$\mathrm{d}\mathbf{x}^k = \frac{\partial\mathbf{x}}{\partial s^k}\mathrm{d}s^k = \mathbf{e}_k\mathrm{d}s^k\,. \tag{9}$$

From these the *directed area element* of a surface $S = \{\mathbf{x}(s^1,s^2)\}$ can be found to be

$$\mathrm{d}^2\mathbf{x} = \mathrm{d}\mathbf{x}^1\wedge\mathrm{d}\mathbf{x}^2 = \mathbf{e}_1\wedge\mathbf{e}_2\mathrm{d}s^1\mathrm{d}s^2\,, \tag{10}$$

and the *directed surface integral* defined in terms of an iterated scalar integral by

$$\int_S F\mathrm{d}^2\mathbf{x} = \int_S F\mathrm{d}\mathbf{x}^1\wedge\mathrm{d}\mathbf{x}^2 = \int_{\alpha_1}^{\beta_1}\mathrm{d}s^1\int_{\alpha_2(s^1)}^{\beta_2(s^1)}\mathrm{d}s^2 F\mathbf{e}_1\wedge\mathbf{e}_2\,. \tag{11}$$

In n dimensions the grade n directed hyper-volume element $\mathrm{d}^n\mathbf{x}$ is denoted by $\mathrm{d}X$ and the grade $(n-1)$ hyper-surface element $\mathrm{d}^{n-1}\mathbf{x}$ by $\mathrm{d}S$.

Because on a closed curve C, directed line elements occur pairwise with opposite signs, it is intuitively clear that

$$\oint_C \mathrm{d}\mathbf{x} = 0\,. \tag{12}$$

One of the most powerful applications of this is a generalization of Stokes' theorem, which relates differentiation and integration on manifolds.

Theorem 1 (Generalized Stokes' Theorem). *Let V be a vector manifold with boundary ∂V, $L(A)$ a multivector-valued function of a multivector argument A. Then*

$$\int_V \dot{L}(\dot{\nabla}\mathrm{d}X) = \oint_{\partial V} L(\mathrm{d}S)\,. \tag{13}$$

We omit the proof here and refer the reader to [2] or [5]. Let us point out, though, that the orientation of the manifold as well as its boundary is taken care of by the definition of the directed measures $\mathrm{d}X$ and $\mathrm{d}S$, respectively.

One important consequence of Stokes' theorem is the fact that the path integral along a closed curve in any gradient field is zero. In order to see this, consider the function $L(A) = \mathbf{f} \cdot A$ with $\mathbf{f}$ and A vectors in a two-dimensional manifold V and $\mathbf{f}$ the gradient of some scalar-valued function w, i.e. $\mathbf{f} = \nabla w = \nabla \wedge w$. Then

$$\begin{aligned}\oint_{\partial V} \mathbf{f} \cdot \mathrm{d}\mathbf{x} &= \int_V \dot{\mathbf{f}} \cdot (\dot{\nabla} \mathrm{d}^2\mathbf{x}) \\ &= \int_V (\dot{\mathbf{f}} \wedge \dot{\nabla}) \cdot \mathrm{d}^2\mathbf{x} \\ &= -\int_V (\nabla \wedge \nabla \wedge w) \cdot \mathrm{d}^2\mathbf{x} = 0\,. \end{aligned} \tag{14}$$

Here, $\mathrm{d}^2\mathbf{x}$ has grade two, serving as a pseudoscalar for the tangent space of V. By basic rewriting rules available in geometric algebra, it can be used to swap the inner and outer product, which we did in line two, above.

3 Fluid Dynamics

Consider a fluid as a mass of particles moving in (n-dimensional) space. As a particle is moved around by external forces and by bouncing off other particles, its position $\mathbf{x}(\mathbf{x}_0,t) = (x^1(x_0^1,\ldots,x_0^n,t),\ldots,x^n(x_0^1,\ldots,x_0^n,t))$ at a certain time t depends on its initial position $\mathbf{x}_0 = (x_0^1,\ldots,x_0^n)$.

An equivalent description of the fluid flow is to represent it as a vector field as it exists at a given time t.

$$\mathbf{u}(\mathbf{x},t) = \frac{\partial \mathbf{x}(\mathbf{x}_0,t)}{\partial t} \tag{15}$$

$$= (u^1(\mathbf{x},t),\ldots,u^n(\mathbf{x},t))\,, \tag{16}$$

where we obtain (16) from (15) by solving for $\mathbf{x}_0$.

However, the particle description has its use and we will let $\varphi_t(\mathbf{x}) = \varphi(\mathbf{x},t)$ denote the position of a particle – initially at position $\mathbf{x}$ – after time t. This implies $\varphi(\mathbf{x},0) = \mathbf{x}$. φ is called the *fluid flow map*.

As a physical entity the flow not only obeys the rules that apply to vector fields in general, but also has to yield to a number of physical principles.

The change of any component of $\mathbf{u}$ with time *implicitly* depends on all the other components, as well as *explicitly* on time. Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(\mathbf{x},t) = (\mathbf{u} \cdot \dot{\nabla})\dot{\mathbf{u}} + \frac{\partial \mathbf{u}}{\partial t}\,, \tag{17}$$

as can be verified by applying the chain rule. We write $\frac{\mathrm{D}}{\mathrm{D}t} = (\mathbf{u} \cdot \nabla) + \frac{\partial}{\partial t}$, an operator which is called the *material derivative*. It takes into acount that the fluid is moving and the particle position changes with time.

The law of *conservation of mass* holds, stating that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0\,, \tag{18}$$

where $\rho = \rho(\mathbf{x},t)$, the scalar-valued *mass density* of the fluid.

There is a position- and time-dependent scalar-valued function $p(\mathbf{x},t)$, *pressure*, which describes a force acting on any arbitrary surface inside the fluid, in the direction of the surface normal.

While pressure is a force acting locally, most real world fluids are subject to global forces such as gravity, acting on the whole volume of fluid. These forces are called *body forces* and denoted $\mathbf{b}$.

The above properties hold for all fluid flows, but there is a number of restrictions that can be placed on a flow to create interesting and important special cases.

An *ideal fluid* is a fluid in which no internal friction occurs. Therefore it does not have viscosity. Note that in nature ideal fluids do not exist. However, they permit a simplification of certain concepts that can be extended later on.

In an ideal fluid, the law of *balance of momentum* holds, stating that

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \rho \mathbf{b}\,. \tag{19}$$

A fluid flow is called *incompressible*, if its mass density is constant following the flow, i.e. $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$. Equivalently, $\nabla \cdot \mathbf{u} = 0$ (see [2] for proof).

A fluid flow is called *homogeneous*, if its mass density is constant in space.

A fluid flow is called *isentropic*, if there is a scalar-valued function $w(\mathbf{x},t)$, called *enthalpy*, with

$$\nabla w = \frac{1}{\rho} \nabla p\,. \tag{20}$$

Figuratively speaking, w is a measure of the *heat content* of a fluid. It consists of the *internal energy* and *pressure work*. Internal energy is made up by molecular rotation and oscillation, potential energy of chemical bonds etc. and roughly proportional to the fluid's temperature. Pressure work is the work that is done to establish the fluid's volume agains pressure's influence trying to compress the fluid.

4 Application of Geometric Algebra to Fluid Dynamics

Definition 1 (Circulation). Given a closed contour C at time $t_0 = 0$, denote with $C_t = \varphi(C,t)$ the contour transported by the flow at time t. Then the *circulation around* C_t is

$$\Gamma_{C_t} = \oint_{C_t} \mathbf{u} \cdot \mathrm{d}\mathbf{x}\,. \tag{21}$$

Theorem 2 (Circulation Theorem). *In an isentropic flow not subject to any body forces,* Γ_{C_t} *is constant in time.*

Proof. Suppose $\mathbf{x}(s), 0 \leq s \leq 1$ is a parametrization of C. Then $\varphi(\mathbf{x}(s),t), 0 \leq s \leq 1$ is a paramtetrization of C_t. Recalling the form of the directed line element we write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{C_t}\mathbf{u}\cdot\mathrm{d}\mathbf{x} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \mathbf{u}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}\mathrm{d}s \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(\varphi_t(\mathbf{x}(s)),t)\cdot\frac{\mathrm{d}}{\mathrm{d}s}\varphi_t(\mathbf{x}(s))\mathrm{d}s \\ &\quad + \int_0^1 \mathbf{u}(\varphi_t(\mathbf{x}(s)),t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\varphi_t(\mathbf{x}(s))\mathrm{d}s \\ &= \int_0^1 \frac{\mathrm{D}\mathbf{u}(\varphi_t(\mathbf{x}(s),t))}{\mathrm{D}t}\cdot\mathrm{d}\mathbf{x} \\ &\quad + \int_0^1 \mathbf{u}(\varphi_t(\mathbf{x}),t)\cdot\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{u}(\varphi_t(\mathbf{x}),t)\mathrm{d}s\,.\end{aligned} \tag{22}$$

But the final term is equal to $\frac{1}{2}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left[\mathbf{u}^2(\varphi_t(\mathbf{x}),t)\right]\mathrm{d}s$, because vectors commute. On the other hand $\int_C \frac{\mathrm{d}}{\mathrm{d}s} f(s)\mathrm{d}s = \int_C \mathrm{d}\mathbf{f} = 0$, if C is closed. So the final term vanishes and we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{C_t}\mathbf{u}\cdot\mathrm{d}\mathbf{x} = \int_{C_t}\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\cdot\mathrm{d}\mathbf{x}\,. \tag{23}$$

Finally, for isentropic flows $\frac{1}{\rho}\nabla p = \nabla w$. Balance of momentum states that $\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p + \rho\mathbf{b}$. Excluding body forces ($\mathbf{b} = 0$), one concludes $\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla w$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_{C_t} = -\int_{C_t}\nabla w\cdot\mathrm{d}\mathbf{x} = 0\,, \tag{24}$$

because the path integral along a closed curve in a gradient field is zero, as was shown in (14). □

Definition 2 (Vortex Sheets, Lines and Tubes). A *vortex sheet* (resp. *vortex line*) is a surface S (resp. a curve C), which – in each of its points – is orthogonal to the *vorticity plane* $\xi \equiv \nabla\wedge\mathbf{u}$ of a fluid flow $\mathbf{u}$.

A *vortex tube* consists of a surface, nowhere locally a vortex sheet, with vortex lines drawn through each point of its bounding curve and extending as far as possible in each direction.

Theorem 3 (Helmholtz' Theorems). *In an isentropic flow without body forces,*

i). if a surface (a curve) moving with the flow is a vortex sheet (a vortex line) at time $t = 0$*, then it remains so for all time.*
ii). if C_1 *and* C_2 *are two closed curves encircling a common vortex tube, then*

$$\int_{C_1}\mathbf{u}\cdot\mathrm{d}\mathbf{x} = \int_{C_2}\mathbf{u}\cdot\mathrm{d}\mathbf{x}. \tag{25}$$

This common value is called the strength *of the vortex tube.*
iii). the strength of a vortex tube is constant in time as the tube moves with the flow.

Proof. Let $S(\mathbf{x})$ be a vortex sheet parametrized by s^1 and s^2. Then $\mathrm{d}S = \frac{\partial S}{\partial s^1}\mathrm{d}s^1 \wedge \frac{\partial S}{\partial s^2}\mathrm{d}s^2 = \mathbf{e}_1(\mathbf{x}) \wedge \mathbf{e}_2(\mathbf{x})\mathrm{d}s^1\mathrm{d}s^2$ is the directed area element of S at $\mathbf{x}$. The orthogonality property between S and ξ can be written as $(\nabla \wedge \mathbf{u}(\mathbf{x})) \cdot (\mathbf{e}_1(\mathbf{x}) \wedge \mathbf{e}_2(\mathbf{x})) = 0$, for all $\mathbf{x}$. Integrating over a vortex sheet and taking the derivative with respect to time yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_S (\nabla \wedge \mathbf{u}) \cdot \mathrm{d}S = \frac{\mathrm{d}}{\mathrm{d}t}\oint_{\partial S} \mathbf{u} \cdot \mathrm{d}\mathbf{x} = 0\,, \tag{26}$$

according to (13), (21) and (24), which proves i) for vortex sheets. The claim for vortex lines is proven by the fact that every vortex line is the intersection of two vortex sheets, locally.

Consider a vortex tube. Let S_1 and S_2 be two arbitrary cross sections of the tube with bounding curves C_1 and C_2, respectively. Let S denote the surface between C_1 and C_2, i.e. the tube's side. Then S_1, S_2 and S enclose a region V and by (13)

$$\begin{aligned}
0 &= \int_V (\nabla \wedge (\nabla \wedge \mathbf{u})) \cdot \mathrm{d}X \\
&= \oint_{S_1 \cup S_2 \cup S} \xi \cdot \mathrm{d}S \\
&= \oint_{S_1} \xi \cdot \mathrm{d}S + \oint_{S_2} \xi \cdot \mathrm{d}S + \oint_S \xi \cdot \mathrm{d}S \\
&= \oint_{S_1} \xi \cdot \mathrm{d}S + \oint_{S_2} \xi \cdot \mathrm{d}S \\
&= \int_{C_1} \mathbf{u} \cdot \mathrm{d}\mathbf{x} + \int_{C_2} \mathbf{u} \cdot \mathrm{d}\mathbf{x}\,,
\end{aligned} \tag{27}$$

because S is a vortex sheet. And since S_1 and S_2 are oriented oppositely, with surface normals pointing "outward", this proves ii). iii) follows from the Circulation Theorem. □

After having shown that geometric algebra can simplify the representation and proof of a number of fluid flow properties, we now present the *stream function* ψ in mulitple dimensions as a graded function, with the grade varying with the dimension of the respective flow. This is possible because of the fact that geometric algebra is a graded algebra and that the grade of an algebraic object can be seen as a variable on a scale, rather than as an immutable, constituting property.

In two dimensions, this function is uniquely determined up to an additive constant. The behaviour of the stream function on streamlines allows for an adjustment of this additive constant, which, in turn, determines the stream function and through it the two-dimensional flow. The lack of such a simple determination is one of the main problems in describing a three-dimensional flow. Finding and exploring constituting properties of the multivector-valued stream function ψ in higher dimensions would greatly facilitate the understanding of general fluid flows.

Employing geometric algebra methods it is easy to define a stream function for n-dimensional, incompressible flows. Assume such a flow with velocity field $\mathbf{u}$, contained in some region D that is simply connected. There exists a function $\psi(\mathbf{x},t)$ of (pure) grade $n-2$, which fulfills

$$\nabla \wedge \psi = \mathbf{u} I_n^{-1}\,, \tag{28}$$

where I_n denotes the unit pseudoscalar in n dimensions. This can be seen by

$$\begin{aligned} 0 &= \nabla \wedge (\nabla \wedge \psi) \\ &= \nabla \wedge (\mathbf{u} I_n^{-1}) \\ &= (\nabla \cdot \mathbf{u}) I_n^{-1}\,, \end{aligned} \tag{29}$$

which is zero if and only if $(\nabla \cdot \mathbf{u})$ is zero, i.e. the flow is incompressible.

Note that (28) does not suffice to determine ψ. For example, adding the curl of any grade $(n-3)$ function $g(\mathbf{x})$ to ψ, such that $\psi' = \psi + \nabla \wedge g$, does not change $\mathbf{u}$.

$$\begin{aligned} \mathbf{u}' &= (\nabla \wedge \psi') I_n \\ &= (\nabla \wedge \psi + \nabla \wedge \nabla \wedge g) I_n \\ &= (\nabla \wedge \psi) I_n \\ &= \mathbf{u}\,. \end{aligned} \tag{30}$$

But, if we fix the divergence of ψ, i.e. $\nabla \cdot \psi = m$, then

$$\begin{aligned} \nabla \cdot \psi' &= \nabla \cdot (\psi + \nabla \wedge g) \\ &= m + \nabla \cdot (\nabla \wedge g) \neq m\,. \end{aligned} \tag{31}$$

Specifying the divergence of ψ is called *choosing a gauge*. Together with (28), choosing a gauge determines ψ. Note that in $n = 2$ dimensions, ψ is a scalar quantity and therefore $\nabla \cdot \psi$ is zero.

A relationship between vorticity ξ and the stream function ψ can be established as follows.

$$\begin{aligned} \xi &= \nabla \wedge \mathbf{u} \\ &= \nabla \wedge ((\nabla \wedge \psi) I_n) \\ &= [\nabla \wedge ((\nabla \wedge \psi) I_n)]\, I_n I_n^{-1} \\ &= [\nabla \cdot ((\nabla \wedge \psi) I_n I_n)]\, I_n^{-1} \\ &= (-1)^{\frac{1}{2}n(n-1)} (\nabla \cdot (\nabla \wedge \psi)) I_n^{-1}\,. \end{aligned} \tag{32}$$

Of special interest is the behaviour of ψ on streamlines.

Definition 3 (Streamlines). At a fixed time, a *streamline* is an integral curve of the velocity field $\mathbf{u}(\mathbf{x},t)$ of a given fluid flow. If $\mathbf{x}(s)$ is a streamline parametrized by s, then $\mathbf{x}(s)$ satisfies

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{u}(\mathbf{x}(s),t), \quad t \text{ fixed}. \tag{33}$$

Let $\mathbf{x}(s)$ be a streamline parametrized by s. Then by (4)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s}\psi(\mathbf{x}(s),t) &= \nabla_s \psi(\mathbf{x}(s),t) \\ &= \dot{\nabla}_s(\dot{\mathbf{x}}(s)\cdot\acute{\nabla}_{\mathbf{x}})\acute{\psi}(\mathbf{x}(s),t) \\ &= \left(\mathbf{u}(\mathbf{x}(s),t)\cdot\acute{\nabla}\right)\acute{\psi}(\mathbf{x}(s),t) \\ &= (\mathbf{u}\cdot\acute{\nabla})\acute{\psi}. \end{aligned} \tag{34}$$

5 Conclusion

In the present work we have shown that the highly complex field of fluid dynamics is accessible by the methods of geometric algebra, which facilitates the understanding of many concepts by – literally – adding new dimensions of consideration.

Also, we have proposed a definition for a stream function in arbitrary-dimensional fluid flows as a graded function. It can be seen to be identical to the known stream function in the special case of two-dimensional flows. We showed how this stream function relates to the geometric algebra formulation of the flow field's curl, i.e. the fluid flow's vorticity.

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