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$G_{6,3}$ GEOMETRIC ALGEBRA

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Abstract. *This paper introduces a new non-Euclidean geometry, which is the generalization of conformal geometry $G_{4,1}$. In this geometry, it is possible to handle not only spheres but also quadric surfaces and their intersections easily as well. The Clifford algebra $G_{6,3}$ is being used as framework, which allows for the creation of a nine dimensional geometry with some new transformations like anisotropic dilatation. This geometry also eases the use of "Quadratic strings"' (the intersection of quadric surfaces) including conics in the $3D$ space.*

1 INTRODUCTION

The use of conformal geometry $G_{4,1}$ allows us to easily resolve geometric problems. Unfortunately, this five dimensional geometry is limited to perform operations between spheres, planes, circles, lines, and points. There are many more problems involving more complex geometric entities, such as quadric surfaces, for example. In this paper, a new geometric framework will be introduced, which allows the use of those geometrics entities. This paper was divided into six sections. The first five sections derive the conformal geometric algebra. The new framework will be covered in section six.

2 THE GEOMETRIC ALGEBRA OF N-D SPACE

In this paper, a geometric algebra $\mathcal{G}_n$ will be specified of the n dimensional space by $\mathcal{G}_{p,q,r}$, where p, q, and r stand for the number of basis vector which squares to 1, -1, and 0 respectively and fulfill $n = p + q + r$.

e_i will be used to denote the vector basis i. In a Geometric algebra $G_{p,q,r}$, the geometric product of two basis vector is defined as

$$e_i e_j = \begin{cases} 1 & for \quad i = j \in 1, \cdots, p \\ -1 & for \quad i = j \in p+1, \cdots, p+q \\ 0 & for \quad i = j \in p+q+1, \cdots, p+q+r. \\ e_i \wedge e_j & for \quad i \neq j \end{cases}$$

This leads to a basis for the entire algebra:

$$\{1\}, \{e_i\}, \{e_i \wedge e_j\}, \ldots, \{e_1 \wedge e_2 \wedge \ldots \wedge e_n\} \tag{1}$$

Any multivector can be expressed in terms of this basis. In the n-D space, there are multivectors of grade 0 (scalars), grade 1 (vectors), grade 2 (bivectors),... up to grade n. Any two such multivectors can be multiplied using the geometric product. Consider two multivectors $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ of grades $\boldsymbol{r}$ and $\boldsymbol{s}$ respectively. The geometric product of $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ can be written as:

$$\boldsymbol{A_r B_s = \langle AB \rangle_{r+s} + \langle AB \rangle_{r+s-2} + \ldots + \langle AB \rangle_{|r-s|}} \tag{2}$$

where $\langle\rangle_{\boldsymbol{t}}$ is used to denote the $\boldsymbol{t}$-grade part of multivector.

$$A_r \cdot B_s = \langle AB \rangle_{|r-s|} \tag{3}$$

$$A_r \wedge B_s = \langle AB \rangle_{|r+s|} \tag{4}$$

consider the geometric product of two vectors $\boldsymbol{ab = \langle ab \rangle_0 + \langle ab \rangle_2 = a \cdot b + a \wedge b}$.

3 THE GEOMETRIC ALGEBRA OF 3D SPACE

The basis for the geometric algebra $\mathcal{G}_{3,0,0}$ of 3D space has $2^3 = 8$ elements and is given by

$$\underbrace{1}_{scalar}, \underbrace{\{e_i, e_j, e_k\}}_{vectors}, \underbrace{\{e_i e_j, e_j e_k, e_k e_i\}}_{bivectors}, \underbrace{\{e_i e_j e_k\} \equiv I}_{trivector}. \tag{5}$$

Since the basis vectors are orthogonal, i.e., $e_i e_j = e_i \cdot e_j + e_i e_j = e_i \wedge e_j$, it is simply written as e_{ij}.

It can easily be verified that the trivector or pseudoscalar $e_ie_je_k$ squares to -1 and commutes with all multivectors in 3D space.

Multiplication of the three basis vectors e_1, e_2, and e_3 by I results in the three basis bivectors $e_1e_2 = Ie_3$, $e_2e_3 = Ie_1$, and $e_3e_1 = Ie_2$. These simple bivectors rotate vectors in their own plane by $90°$, e.g., $(e_1e_2)e_2 = e_1$, $(e_2e_3)e_2 = -e_3$, etc. Identifying the **i**, **j**, **k** of the quaternion algebra with $Ie_1, -Ie_2, Ie_3$, the famous Hamilton relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ can be recovered. Since the **i**, **j**, **k** are bivectors, it comes as no surprise that they represent $90°$ rotations in orthogonal directions and provide a well-suited system for the representation of general 3D rotations.

4 CONFORMAL GEOMETRY

Geometric algebra $G_{4,1} = G_{4,1,0}$ can be used to treat conformal geometry in a very elegant way. To see how this is possible, the same formulation presented in [2] is followed, and the Euclidean vector space $\Re^3$ is represented in $\Re^{4,1}$. This space has an orthonormal vector basis given by $\{e_i\}$. $e_{ij} = e_i \wedge e_j$ are bivectorial bases. Bivector basis e_{23}, e_{31}, and e_{12} correspond together with 1 to Hamilton's quaternions. The Euclidean pseudo-scalar unit $I_e := e_1 \wedge e_2 \wedge e_3$, a pseudo-scalar $I = I_eE$, and the bivector $E := e_4 \wedge e_5 = e_4e_5$ are used for computing Euclidean and conformal duals of multivectors. For more about conformal geometric algebra, refer to [1].

4.1 The Stereographic Projection

Conformal geometry is related to a stereographic projection in Euclidean space. A stereographic projection consists on mapping the points lying on a hypersphere to points lying on a hyperplane. In this case, the projection plane passes through the equator, and the sphere is centered at the origin. To make a projection, a line is drawn from the north pole to each point on the sphere and the intersection of this line, where with the projection plane constitutes the stereographic projection.

For simplicity, the equivalence between stereographic projections and conformal geometric algebra of $\Re^1$ will be illustrated. This paper will focus on work done in $\Re^{2,1}$ with the basis vectors $\{e_1, e_4, e_5\}$ having the above mentioned properties. The projection plane will be the x-axis, and the sphere will be centered at the origin with a unitary radius.

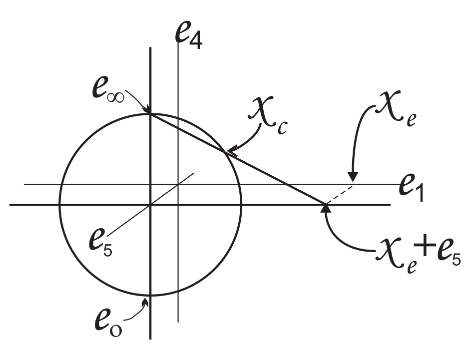

Figure 1: Stereographic projection for 1-D.

Given a scalar x_e representing a point on the x-axis, point x_c, lying on the circle that projects to it, is calculated (see Figure 1). The equation of the line passing through the north pole and x_e is given as $f(x) = -\frac{1}{x_e}x + 1$, and the equation of the circle is $x^2 + g(x)^2 = 1$. Substituting the equation of the line on the circle $g = f$, the point of intersection x_c is obtained, which can be represented in homogeneous coordinates as the vector

$$x_c = 2\frac{x_e}{x_e^2+1}e_1 + \frac{x_e^2-1}{x_e^2+1}e_4 + e_5. \tag{6}$$

From (6) the coordinates on the circle for the point at infinity can be inferred as

$$\begin{aligned} e_\infty &= \lim_{x_e\to\infty}\{x_c\} \\ &= \lim_{x_e\to\infty}\left\{2\frac{x_e}{x_e^2+1}e_1 + \frac{x_e^2-1}{x_e^2+1}e_4 + e_5\right\} \\ &= e_4 + e_5, \qquad (7) \\ e_o &= \frac{1}{2}\lim_{x_e\to 0}\{x_c\} \\ &= \frac{1}{2}\lim_{x_e\to 0}\left\{2\frac{x_e}{x_e^2+1}e_1 + \frac{x_e^2-1}{x_e^2+1}e_4 + e_5\right\} \\ &= \frac{1}{2}(-e_4 + e_5), \qquad (8) \end{aligned}$$

Note that (6) can be rewritten as

$$x_c = x_e + \frac{1}{2}x_e^2 e_\infty + e_o, \tag{9}$$

4.2 Spheres and Planes

The equation of a sphere of radius ρ centered at point $p_e \in \Re^3$ can be written as $(x_e - p_e)^2 = \rho^2$. Since $x_c \cdot y_c = -\frac{1}{2}(\mathtt{x}_e - \mathtt{y}_e)^2$, where $\mathtt{x}_e$ and $\mathtt{y}_e$ are the Euclidean components, and $x_c \cdot p_c = -\frac{1}{2}\rho^2$, the formula above can be rewritten in terms of homogeneous coordinates. Since $x_c \cdot e_\infty = -1$, the expression above can be factored to

$$x_c \cdot (p_c - \frac{1}{2}\rho^2 e_\infty) = 0, \tag{10}$$

This equation corresponds to the so called Inner Product Null Space (IPNS) representation, which finally yields the simplified equation for the sphere as $s = p_c - \frac{1}{2}\rho^2 e_\infty$. Note from this equation that a point is just a sphere with a radius of zero. Alternatively, the dual of the sphere is represented as 4-vector $s^* = sI$. The advantage of the dual form is that the sphere can be directly computed from four points as

$$s^* = x_{c_1} \wedge x_{c_2} \wedge x_{c_3} \wedge x_{c_4}. \tag{11}$$

If one of these points are replaced for the point at infinity, the equation of a 3D plane is obtained

$$\pi^* = x_{c_1} \wedge x_{c_2} \wedge x_{c_3} \wedge e_\infty. \tag{12}$$

So that π is put in standard IPNS form

$$\pi = I\pi^* = n + de_\infty \tag{13}$$

Where n is the normal vector and d represents the Hesse distance for the 3D space.

4.3 Circles and Lines

A circle z can be regarded as the intersection of two spheres s_1 and s_2 as $z = (s_1 \wedge s_2)$ in IPNS. The dual form of the circle can be expressed by three points lying on the circle, namely

$$z^* = x_{c_1} \wedge x_{c_2} \wedge x_{c_3}. \tag{14}$$

Similar to the case of planes, lines can be defined by circles passing through the point at infinity as:

$$L^* = x_{c_1} \wedge x_{c_2} \wedge e_\infty. \tag{15}$$

The standard IPNS form of the line can be expressed as

$$L = \mathbf{n}I_e - e_\infty \mathbf{m} I_e, \tag{16}$$

where **n** and **m** stand for the line orientation and moment, respectively. The line in the IPNS standard form is a bivector representing the six Plücker coordinates.

Table 1: Representation of conformal geometric entities

Entity	IPNS Representation	OPNS Dual representation
Sphere	$s = p - \frac{1}{2}\rho^2 e_\infty$	$s^* = x_1 \wedge x_2 \wedge x_3 \wedge x_4$
Point	$x_c = x_e + \frac{1}{2}x_e^2 e_\infty + e_0$	$x^* = s_1 \wedge s_2 \wedge s_3 \wedge s_4$
Line	$L = \mathbf{n}I_e - e_\infty \mathbf{m} I_e$	$L^* = x_1 \wedge x_2 \wedge e_\infty$
Plane	$\pi = n + de_\infty$	$\pi^* = x_1 \wedge x_2 \wedge x_3 \wedge e_\infty$
Circle	$z = s_1 \wedge s_2$	$z^* = x_1 \wedge x_2 \wedge x_3$
Pair of Points	$P_p = s_1 \wedge s_2 \wedge s_3$	$P_p^* = x_1 \wedge x_2$

5 RIGID TRANSFORMATIONS

Rigid transformations can be expressed in conformal geometry carrying out plane reflections.

5.0.1 Reflection

The combination of reflections of conformal geometric entities enables the forming of other transformations. The reflection of a point x with respect to the plane π is equal to x minus twice the directed distance between the point and plane (see the Figure 2). That is, $ref(x) = x - 2(\pi \cdot x)\pi^{-1}$. This expression is calculated by using the reflection $ref(x_c) = -\pi x_c \pi^{-1}$ and the Clifford product of vectors property $2(b \cdot a) = ab + ba$.

For a IPNS geometric entity Q, the reflection with respect to the plane π is given as

$$Q' = \pi Q \pi^{-1} \tag{17}$$

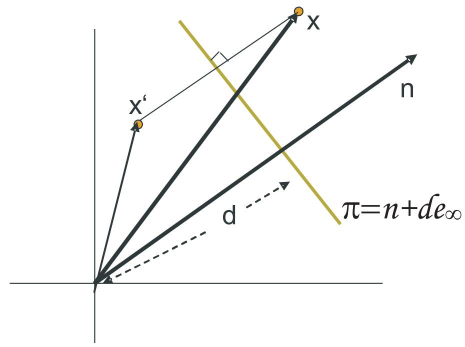

Figure 2: Reflection of a point x with respect to the plane π.

5.0.2 Translation

The translation of conformal geometric entities can be done by carrying out two reflections at parallel planes π_1 and π_2 (see Figure 3). That is

$$Q' = \underbrace{(\pi_2\pi_1)}_{T_a} Q \underbrace{(\pi_1^{-1}\pi_2^{-1})}_{\widetilde{T_a}} \tag{18}$$

$$T_a = (n + de_\infty)n = 1 + \frac{1}{2}ae_\infty = e^{\frac{a}{2}e_\infty} \tag{19}$$

With $a = 2dn$.

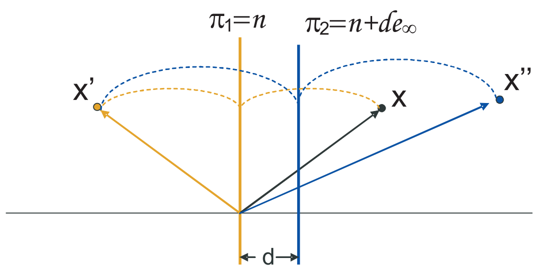

Figure 3: Reflection about parallel planes.

5.0.3 Rotation

The rotation is the product of two reflections at nonparallel planes which pass through the origin (see Figure 4)

$$Q' = \underbrace{(\pi_2\pi_1)}_{R_\theta} Q \underbrace{(\pi_1^{-1}\pi_2^{-1})}_{R_\theta} \tag{20}$$

Or the conformal product computation of the normals of the planes.

$$R_\theta = n_2 n_1 = cos(\frac{\theta}{2}) - sin(\frac{\theta}{2})l = e^{-\frac{\theta}{2}l} \tag{21}$$

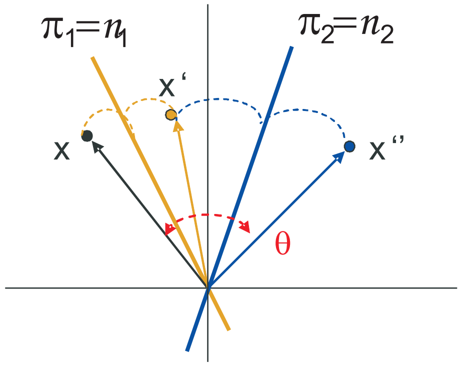

Figure 4: Reflection about nonparallel planes.

With $l = n_2 \wedge n_1$ and θ twice the angle between the planes π_2 and π_1. The screw motion *called motor* is related to an arbitrary axis L is $M = TR\widetilde{T}$

$$Q' = \underbrace{(TR\widetilde{T})}_{M_\theta} Q \underbrace{(T\widetilde{R}\widetilde{T})}_{\widetilde{M_\theta}} \tag{22}$$

$$M_\theta = TR\widetilde{T} = cos(\frac{\theta}{2}) - sin(\frac{\theta}{2})L = e^{-\frac{\theta}{2}L} \tag{23}$$

6 $G_{6,3}$ GEOMETRY

In this secction, the $G_{6,3}$ Geometry will be introduced. By definition, this is a 9-dimensional geometry, which has six basis squaring to 1 and three basis squaring to -1 defined as follows

$$\begin{aligned} e_1^2, \cdots, e_6^2 &= 1 \\ e_7^2, \cdots, e_9^2 &= -1 \end{aligned} \tag{24}$$

Similarly to Conformal geometry ($G_{4,1}$), the stereographic projection will be used to map points of $\Re^1$ into points on $\Re^{2,1}$. The main difference is that the stereographic projection will be performed independently for each axis. This means basis e_4 and e_7 are being used for axis e_1, basis e_5 and e_8 for e_2, and basis e_6 and e_9 for e_3. Since this geometry allows the manipulation of quadric entities, the points mapped to this space will be denoted by using subindex Q. Taking a point in the Euclidean space $(x, y, z) \in \Re^3$ will be mapped on $G_{6,3}$ as follows

$$\begin{aligned} x_Q &= 2\frac{x}{x^2+1}e_1 + \frac{x^2-1}{x^2+1}e_4 + e_7 \\ &+ 2\frac{y}{y^2+1}e_2 + \frac{y^2-1}{y^2+1}e_5 + e_8 \\ &+ 2\frac{z}{z^2+1}e_3 + \frac{z^2-1}{z^2+1}e_6 + e_9. \end{aligned} \tag{25}$$

The point at infinity is being computed by applying the limit $\lim_{x,y,z\to\infty}$ to the point X_Q

$$\begin{aligned} e_\infty &= \lim_{x,y,z\to\infty} \{x_Q\} \\ &= (e_4 + e_7) + (e_5 + e_8) + (e_6 + e_9). \end{aligned} \tag{26}$$

In order to easily handle those vectors, the definition of vectors $e_{\infty x}, e_{\infty y}, e_{\infty z}$ are established as

$$e_{\infty x} = (e_4 + e_7), \tag{27}$$
$$e_{\infty y} = (e_5 + e_8), \tag{28}$$
$$e_{\infty z} = (e_6 + e_9). \tag{29}$$

The introduction of those definitions allows for the equation of the point at infinity to be rewritten (26).

$$e_{\infty} = \frac{1}{3}(e_{\infty x} + e_{\infty y} + e_{\infty z}). \tag{30}$$

Note that e_{∞} was divided by three, since it does not change the meaning of the point at infinity but allows the normalization of the product. Following the same procedure for a point at infinity, the vector representing the origin is computed as

$$\begin{aligned} e_o &= \lim_{x,y,z\rightarrow 0}\{x_Q\} & (31)\\ &= \frac{1}{2}(-e_4 + e_7) + \frac{1}{2}(-e_5 + e_8) + \frac{1}{2}(-e_6 + e_9). \end{aligned}$$

Similarly new vectors are introduced to describe origins

$$e_{ox} = \frac{1}{2}(-e_4 + e_7), \tag{32}$$
$$e_{oy} = \frac{1}{2}(-e_5 + e_8), \tag{33}$$
$$e_{oz} = \frac{1}{2}(-e_6 + e_9). \tag{34}$$

This way, vector e_0 is given by

$$e_o = (e_{ox} + e_{oy} + e_{oz}). \tag{35}$$

Each one of those new vectors are actually nilpotent, which means their magnitude is equal to zero

$$e_{\infty x}^2 = 0 \quad e_{ox}^2 = 0 \tag{36}$$
$$e_{\infty y}^2 = 0 \quad e_{ox}^2 = 0 \tag{37}$$
$$e_{\infty z}^2 = 0 \quad e_{ox}^2 = 0 \tag{38}$$
$$e_{\infty}^2 = 0 \quad e_o^2 = 0 \tag{39}$$

Additionally, the dot product between these new bases is given by

$$e_{\infty x} \cdot e_{ox} = -1 \quad e_{\infty x} \cdot e_{oz} = 0 \tag{40}$$
$$e_{\infty y} \cdot e_{oy} = -1 \quad e_{\infty y} \cdot e_{oz} = 0 \tag{41}$$
$$e_{\infty z} \cdot e_{oz} = -1 \quad e_{\infty x} \cdot e_{oy} = 0 \tag{42}$$
$$e_{\infty} \cdot e_o = -1 \tag{43}$$

Using these new vectors, equation 25 is rewritten as

$$x_Q = xe_1 + ye_2 + ze_3 + \frac{1}{2}(x^2 e_{\infty x} + y^2 e_{\infty y} + z^2 e_{\infty z}) + e_o \tag{44}$$

This equation represents the mapping to the new $G_{6,3}$ geometry and a generalization of the conformal mapping.

6.1 Geometric entities

In $G_{6,3}$, the basic entity is a quadric surface. Every 2D surface, such as ellipsoids, hyperboloid, spheres etc. could be represented by using vectors grade one. The first geometric entity described here is the ellipsoid because the rest of the quadric surfaces are very similar and could be generated by modifying the ellipsoid.

6.1.1 Ellipsoid

The equation of a standard axis-aligned ellipsoid with center at (h, k, l) and radius (a, b, c) in an xyz-Cartesian coordinate system is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{c^2} = 1 \tag{45}$$

Expanding and reordering

$$\frac{hx}{a^2} + \frac{ky}{b^2} + \frac{lz}{c^2} - \frac{1}{2}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2} - 1\right) - \frac{1}{2}\left(\frac{x^2}{2a^2} + \frac{y^2}{2b^2} + \frac{z^2}{2c^2}\right) = 0. \tag{46}$$

This equation could be rewritten as a dot product of two vectors

$$x_Q \cdot H = 0. \tag{47}$$

where x_Q represents the mapping into $G_{6,3}$ of a euclidean point (x, y, z) (recalling equation (44)), and H represents the ellipsoid as a vector and is given by

$$H = \frac{h}{a^2}e_1 + \frac{k}{b^2}e_2 + \frac{l}{c^2}e_3 + \frac{1}{2}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} + \frac{l^2}{c^2} - 1\right)e_\infty + \frac{1}{a^2}e_{ox} + \frac{1}{b^2}e_{oy} + \frac{1}{c^2}e_{oz}. \tag{48}$$

The ellipsoid H is a grade one vector. Alternatively, it is possible to create an ellipsoid using six points in a general configuration lying on an ellipsoid

$$H^* = x_{Q1} \wedge x_{Q2} \wedge x_{Q3} \wedge x_{Q4} \wedge x_{Q5} \wedge x_{Q6} \tag{49}$$

where $*$ denotes dual or OPNS, here the ellipsoid H^* is grade six. Equation 49 is not only valid for an ellipsoid, but could also be utilized in the creation of a hyperboloid (image 5), paraboloid, or any axis aligned quadric, in general.

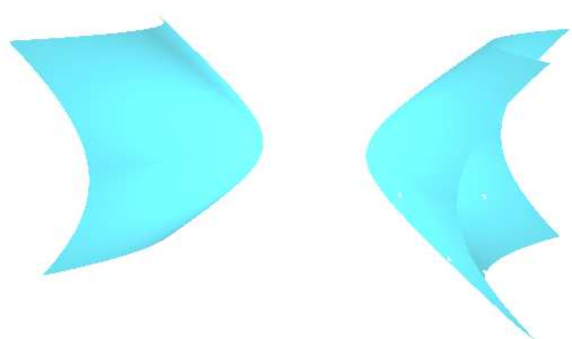

Figure 5: hyperboloid generated by wedging six points.

Image 6 shows an ellipsoid covered with many points on the surface. These points are the result of a genetic algorithm performed to evaluate the condition (47) and empirically determine the kind of surface vector H is representing (in order to validate the theory).

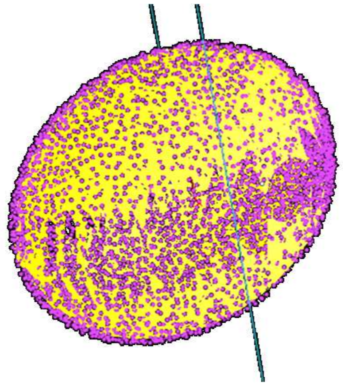

Figure 6: Ellipsoid generated by evaluation of $X_Q \cdot H = 0$.

6.1.2 Sphere

Evaluating the equation of ellipsoid 48 using the same radio r for axis x,y and z, that is $r = a = b = c$

$$S = \frac{h}{r^2}e_1 + \frac{k}{r^2}e_2 + \frac{l}{r^2}e_3 + \frac{1}{2}\left(\frac{h^2}{r^2} + \frac{k^2}{r^2} + \frac{l^2}{r^2} - 1\right)e_\infty + \frac{1}{r^2}e_{ox} + \frac{1}{r^2}e_{oy} + \frac{1}{r^2}e_{oz}. \tag{50}$$

since homogeneous coordinates are being used, the vectors are equal up to scalar factor, so it is not affected by multiplying it by r^2

$$\begin{aligned} S = he_1 + ke_2 + le_3 + \frac{1}{2}\left(h^2 + k^2 + l^2 - r^2\right)e_\infty \\ +e_{ox} + e_{oy} + e_{oz}. \end{aligned} \tag{51}$$

Recalling equation 35 and introducing $p = he_1 + ke_2 + le_3$ representing the center of the sphere,

$$S = p + \frac{1}{2}\left(p^2 - r^2\right)e_\infty + e_o. \tag{52}$$

which is the typical equation used for the sphere in conformal geometry

6.1.3 Cylinder

The degenerated quadric surfaces are also vectors. Starting from the ellipsoid vector 48 and considering infinity as the limit for one of the radios, for example G=$\lim_{c\to\infty} H$

$$G = \frac{h}{a^2}e_1 + \frac{k}{b^2}e_2 + \frac{1}{2}\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right)e_\infty + \frac{1}{a^2}e_{ox} + \frac{1}{b^2}e_{oy}$$

In this case, G represents a cylinder aligned to z axis, and the equation $G \cdot X_Q = 0$ describes a cylinder. It is also possible to create a cylinder replacing a point in the ellipsoid by a point at infinity

$$G^* = x_{Q1} \wedge x_{Q2} \wedge x_{Q3} \wedge x_{Q4} \wedge x_{Q5} \wedge e_{\infty z} \tag{53}$$

since the ”ellipsoid” ”touches” the point at infinity $e_{\infty z}$, G^* represents the dual form of a cylinder aligned to the z axis.

6.1.4 Pair of parallel planes

Starting from ellipsoid vector 48 and considering infinity as the limit for two of the radios, for example G=$\lim_{b,c\to\infty} H$

$$\begin{aligned} G &= \frac{h}{a^2}e_1 + \frac{1}{2}\left(\frac{h^2}{a^2} - 1\right)e_\infty + \frac{1}{a^2}e_{ox} \\ G &= he_1 + \frac{1}{2}\left(h^2 - a^2\right)e_\infty + e_{ox} \end{aligned} \tag{54}$$

Then $x_Q \cdot G = 0$ only if

$$(x-h)^2 = a^2 \tag{55}$$

which is the equation of two parallel planes to the yz plane. In addition, this entity could be generated by using

$$\pi^* = x_{Q1} \wedge x_{Q2} \wedge x_{Q3} \wedge x_{Q4} \wedge e_{\infty y} \wedge e_{\infty z} \tag{56}$$

6.1.5 plane

Similarly to conformal geometry the equation of the plane is given by

$$\pi = n + de_\infty. \tag{57}$$

where $n = n_x e_1 + n_y e_2 + n_z e_3$ is a vector representing the normal of the plane, and d is the Hesse distance. Alternatively, it is possible to create a plane by wedging three points in a general configuration (not aligned) and three of points at infinity

$$\pi^* = x_{Q1} \wedge x_{Q2} \wedge x_{Q3} \wedge e_{\infty x} \wedge e_{\infty y} \wedge e_{\infty z} \tag{58}$$

6.1.6 Pair of non-parallel planes

This is also a degenerated quadric (see image 7), and it consist in a pair of planes intersecting each other. On $G_{6,3}$ this entity is bein represented by a vector

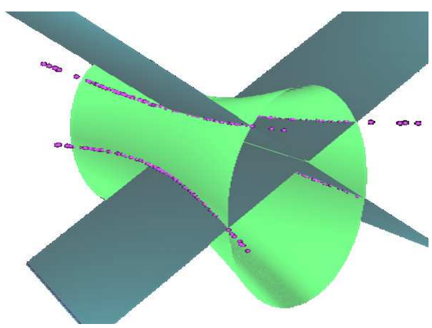

Figure 7: A degenerated quadric surface intersection.

$$G = \frac{h}{a^2}e_1 - \frac{k}{b^2}e_2 + \frac{1}{2}\left(\frac{h^2}{a^2} - \frac{k^2}{b^2}\right)e_\infty + \frac{1}{a^2}e_{ox} - \frac{1}{b^2}e_{oy} \tag{59}$$

Then $x_Q \cdot G = 0$ only if

$$\frac{(x-h)^2}{a^2} = \frac{(y-k)^2}{b^2} \tag{60}$$

This equation represents a pair of planes intersected at (h,k) given by $y = k \pm \frac{b}{a}(x-h)$

6.1.7 ”Quadratic Strings”

As with conformal geometric algebra, $G_{6,3}$ geometry eases the computation of the intersection of geometric entities; it is possible, for example, computing the intersection between a paraboloid and one ellipsoid. By doing this, it is possible to generate one dimensional geometric entities such as conics (cone/plane intersection). If H and G represent quadric surfaces, the intersection S (see picture 8) could be computed as simply as

$$S = H \wedge G \tag{61}$$

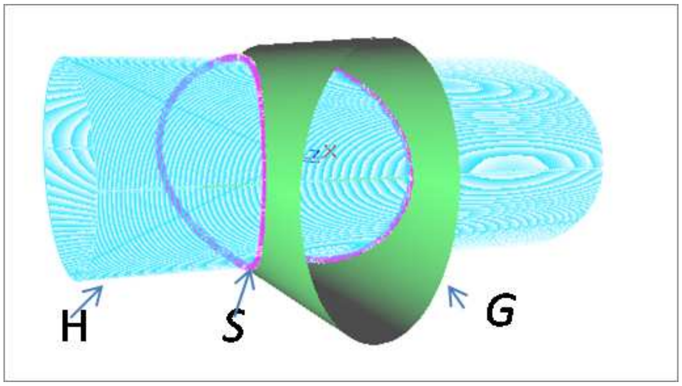

Figure 8: Quadric surfaces intersection.

Where S represents a one dimensional surface and is described by a Bivector, points x_Q lying in the intersection accomplish $x_Q \cdot S = 0$. The line is also a ”string” and is being represented by a bivector that could be generated by using plucker coordinates as

$$L = ne_{\infty} + l \tag{62}$$

where l is a bi-vector representing the direction of the line. it is also possible to generate a line by using two points

$$L^* = x_{Q1} \wedge x_{Q2} \wedge e_{\infty x} \wedge e_{\infty y} \wedge e_{\infty z} \tag{63}$$

In this case, the line is a five grade homogeneous multivector and contains five points: two points x_{Q1} and x_{Q2} and three points at infinity. Image 9 shows the render of a bivector generated by the intersection of two quadric surfaces

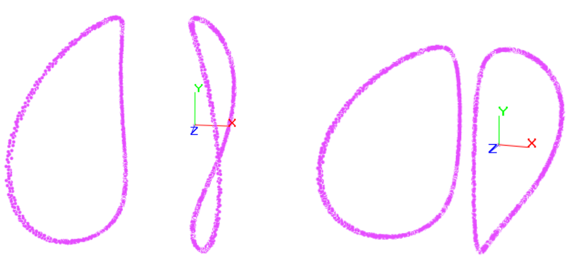

Figure 9: 1-D quadric surfaces (”Quadric strings”).

6.1.8 Zero dimensional entities

These entities could be a point, a pair of points, three, or even four points in space. They represent the intersection between a Quadric surface and a conic or the wedging of four points.

$$4P^* = x_{Q1} \wedge x_{Q2} \wedge x_{Q3} \wedge x_{Q4} \tag{64}$$

6.2 Transformations

6.2.1 Anisotropic Dilatation

Similarly to the rotation and translation, the dilation could be seen as a reflection. In this case, however, between two concentric spheres (see image 10)

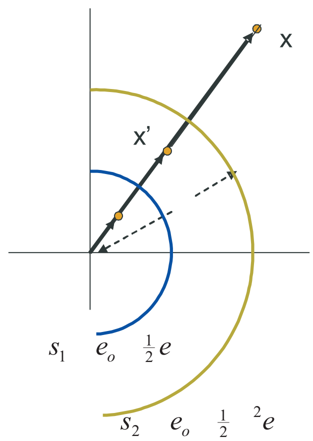

Figure 10: Reflection between concentric spheres.

$$D_x = (e_{ox} - \frac{1}{2}e_{\infty x})(e_{ox} - \frac{1}{2}\rho^2 e_{\infty x}) \tag{65}$$

$$D_x = \frac{1}{2}(1 - e_{47}) + \frac{1}{2}(1 + e_{47})\rho^2 \tag{66}$$

$$D_x = \frac{1}{2}\rho^{-1}(1 - e_{47}) + \frac{1}{2}(1 + e_{47})\rho = e^{e_{47}\phi} \tag{67}$$

where $\phi = \ln(\rho)$. Then $D_x = e^{e_{47}\phi}$ dilates every geometric entity only in the x axis. In the same way, $D_y = e^{e_{58}\phi}$ and $D_z = e^{e_{69}\phi}$ dilates y and z axis, respectively. For example, assuming S as a sphere, the operation will generate an ellipsoid S'.

$$S' = D_z D_y D_x S \widetilde{D_x}\widetilde{D_y}\widetilde{D_z} \tag{68}$$

6.3 Pseudo-scalar

In this paper, two descriptions for every geometric entity were used (standard and dual* or IPNS and OPNS respectively). It is possible and very useful to move from one to other. In order to do that, some new will pseudo scalars be defined

$$I_e = e_1 \wedge e_2 \wedge e_3 \tag{69}$$

$$I_{ds} = Ie \wedge e_{\infty x} \wedge e_{\infty y} \wedge e_{\infty z} \wedge e_o \tag{70}$$

$$I_{sd} = Ie \wedge e_{ox} \wedge e_{oy} \wedge e_{oz} \wedge e_\infty \tag{71}$$

Those pseudo-scalars allow us to move an entity from OPNS to IPNS, and it is valid for 2D, 1D, and 0D quadratic surfaces.

$$H = H^* \cdot I_{ds} \tag{72}$$
$$H^* = H \cdot I_{sd} \tag{73}$$

7 CONCLUSION

One of the main advantages of this geometry is the fact that it is possible to handle many more geometric entities as vectors, and it provides a framework to perform operations between quadric surfaces and their intersections. It also provides an easy way to represent 1-dimensional quadric surfaces. Instead of using a bundle of parametric equations, they are given by a bi-vector. This geometry also provides a good framework to perform any conformal transformation and allows anisotropic transformations like, for example, anisotropic dilatation.

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