# Geometric Algebra Computing

## 3D calculations

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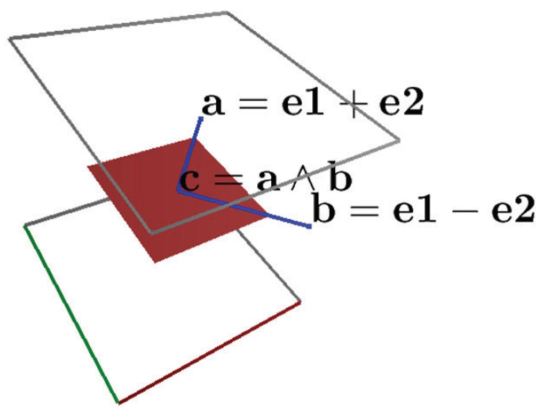


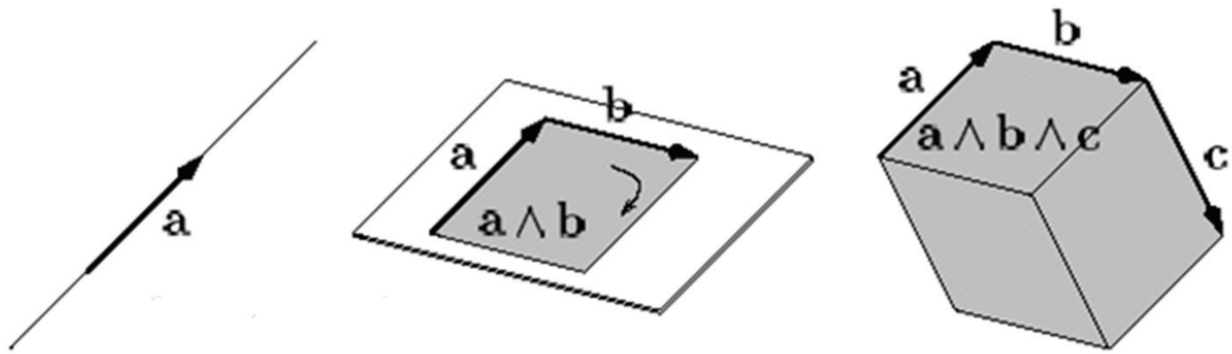
# Calculations in 3D Euclidean GA

# The blades of 3D euclidean geometric algebra

| | blade | grade | abbreviation |
|---|---|---|---|
| 1. | $1$ | 0 | 1 |
| 2. | $e_1$ | 1 | e1 |
| 3. | $e_2$ | 1 | e2 |
| 4. | $e_3$ | 1 | e3 |
| 5. | $e_2 \wedge e_3$ | 2 | e23 |
| 6. | $e_3 \wedge e_1$ | 2 | e31 |
| 7. | $e_1 \wedge e_2$ | 2 | e12 |
| 8. | $e_1 \wedge e_2 \wedge e_3$ | 3 | I |

# The main products of geometric algebra

- Outer Product




vector bivector trivector

- Inner Product

- Geometric Product

# The geometric product of 2 basis vectors

$$\text{geometric algebra } G_{p,q}$$

$$with \quad n = p + q$$

$$\rightarrow$$

$$define$$

$$e_i e_j = \begin{cases} 1 & for \quad i = j \in \{1, \ldots, p\} \\ -1 & for \quad i = j \in \{p+1, \ldots, n\} \\ e_{ij} = e_i \wedge e_j = -e_j \wedge e_i & for \quad i \neq j \end{cases}$$

# Properties of the outer product

| | Property | Meaning |
|---|---|---|
| 1. | Anticommutativity | $u \wedge v = -(v \wedge u)$ |
| 2. | Distributivity | $u \wedge (v + w) = u \wedge v + u \wedge w$ |
| 3. | Associativity | $u \wedge (v \wedge w) = (u \wedge v) \wedge w$ |

What is $\mathbf{a} \wedge \mathbf{a}$ then ?

$a \wedge a = -(a \wedge a)$

$\Rightarrow a \wedge a = 0$

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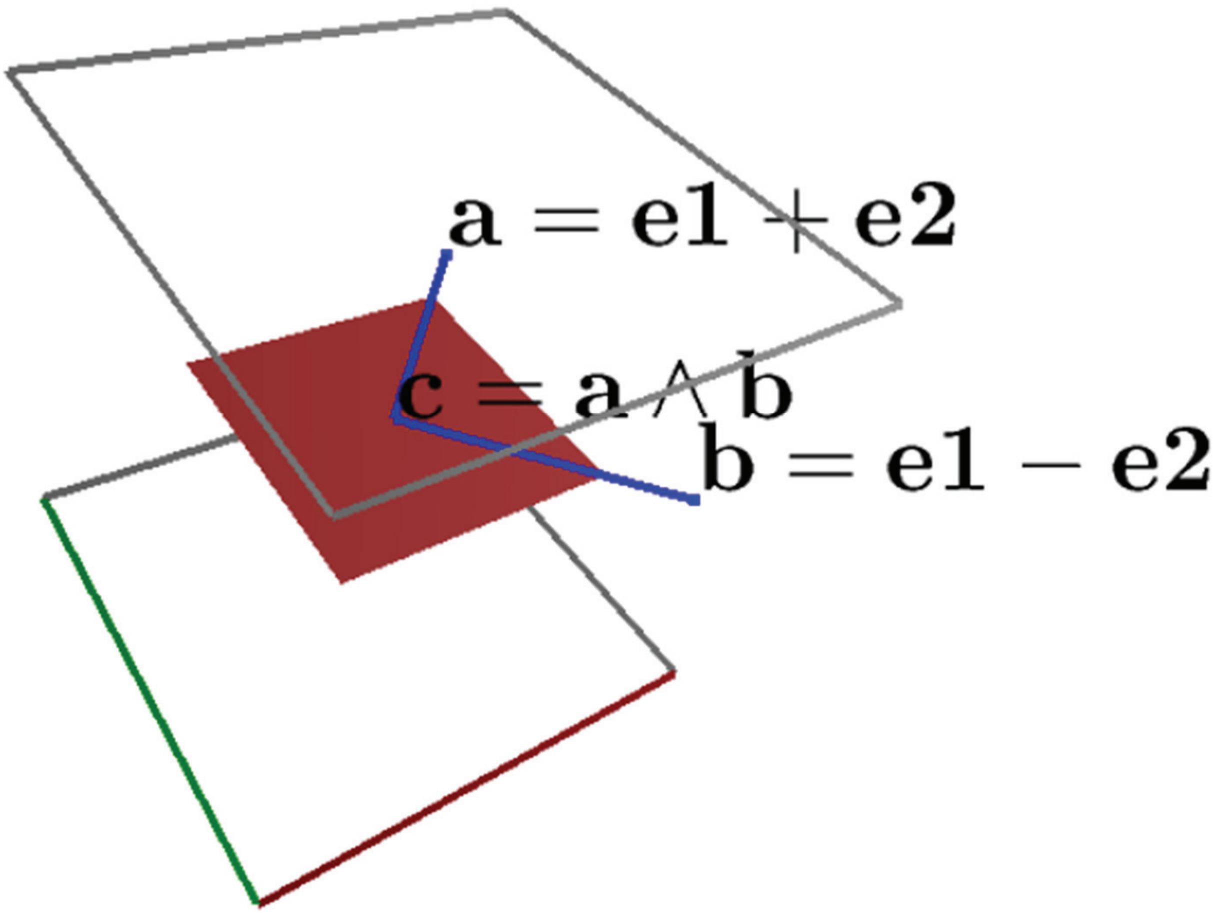
What is $\mathbf{a} \wedge \mathbf{a}$ then ?

$$a \wedge a = -(a \wedge a) = 0$$

- Note: the outer product can be used as a measure of parallelness

# Example bivectorE3.clu

```
DefVarsE3();
:Blue;
:a = e1 + e2;
:b = e1 - e2;
:Red;
:c = a ^ b;
```



# Computation example 1 (page 19)

$$c = a \wedge b = (e_1 + e_2) \wedge (e_1 - e_2) \quad | \text{ Distributivity}$$

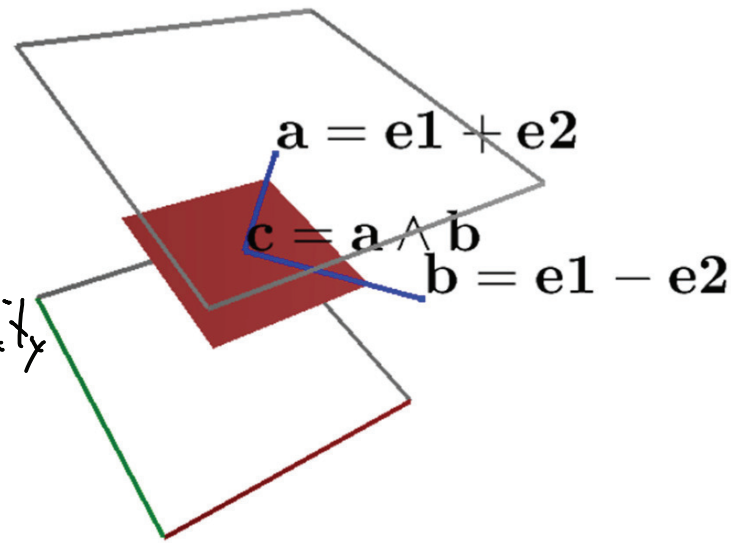
$$= \underbrace{(e_1 \wedge e_1)}_{0} - (e_1 \wedge e_2) + (e_2 \wedge e_1) - \underbrace{(e_2 \wedge e_2)}_{0}$$

$$= -(e_1 \wedge e_2) + (e_2 \wedge e_1)$$

$$= (e_2 \wedge e_1) + (e_2 \wedge e_1) \quad | \text{ anti-commutativity}$$

$$= 2(e_2 \wedge e_1) = 2 e_{21}$$

$$= -2 e_{12}$$

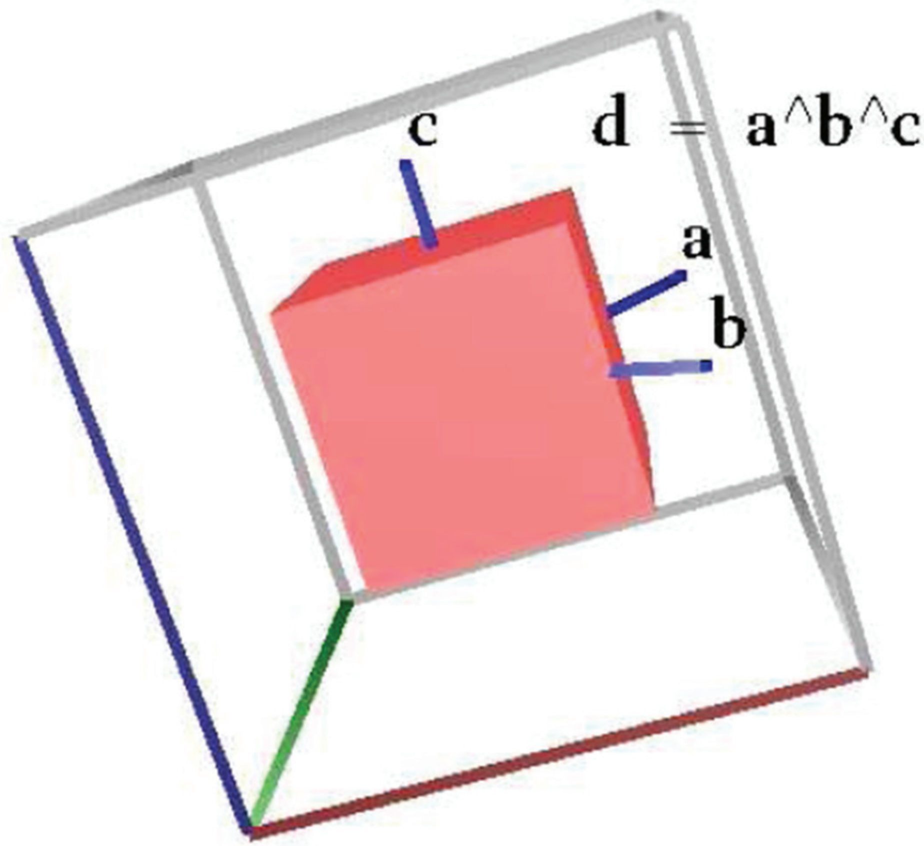


# trivectorE3.clu

```
DefVarsE3();

:Blue;
:a = e1 + e2;
:b = e1 - e2;
:c = e3;

:Red;
:d = a ^ b ^ c;
?d;
```



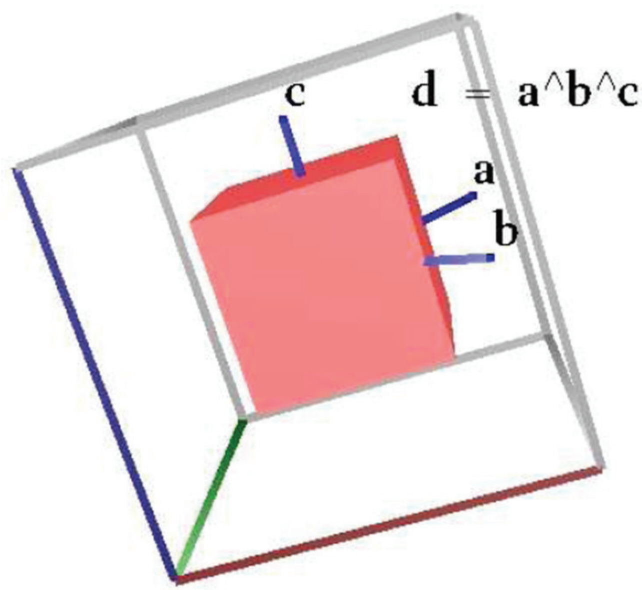
# trivectorE3.clu (page 19)

d=a^b^c $= (e_1 + e_2) \wedge (e_1 - e_2) \wedge e_3$

$= 2(e_2 \wedge e_1) \wedge e_3 \quad = 2\, \underbrace{e_2 \wedge e_1}_{-e_1 \wedge e_2} \wedge e_3$

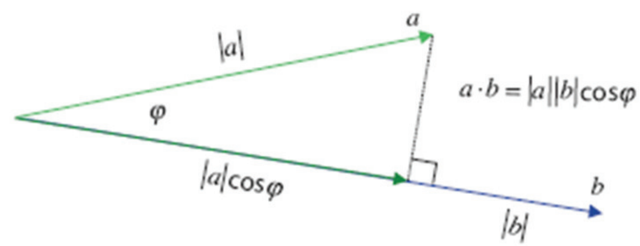
$= -2\, e_1 \wedge e_2 \wedge e_3$

$= -2\, e_{123} = -2\, I$



# The inner product of two vectors

- Inner product = Scalar product
  is true only for vectors!




- For vector and bivector:

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, then the bivector $\mathbf{b} \wedge \mathbf{c} \in \mathcal{C}\ell(\mathbb{R}^n)$. The inner product of $\mathbf{a}$ with this bivector gives,

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b}. \tag{1.9}$$

- General rule in [Perwass/Hildenbrand] page 6

# Reverse, norm of subspaces

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$$\| A \| = \sqrt{\tilde{A} \cdot A}$$

*with*

$$\tilde{A} = a_k \wedge \ldots \wedge a_2 \wedge a_1$$

Example:

```
c  = -2 e12
```

In CLUCalc:

```
?reverse = ~c;
```

The reverse of $c$ is computed.
It results in

```
reverse  = 2 e12
```

since the reverse of a blade simply reverses its order.

# The inner product and perpendicularity

As in vector algebra, the result of taking the inner product of two basis vectors,

```
DefVarsE3();
?InnerProduct = e1.e2;
```

is

```
InnerProduct = 0,
```

since the two vectors are **perpendicular** to each other.

# The general inner product

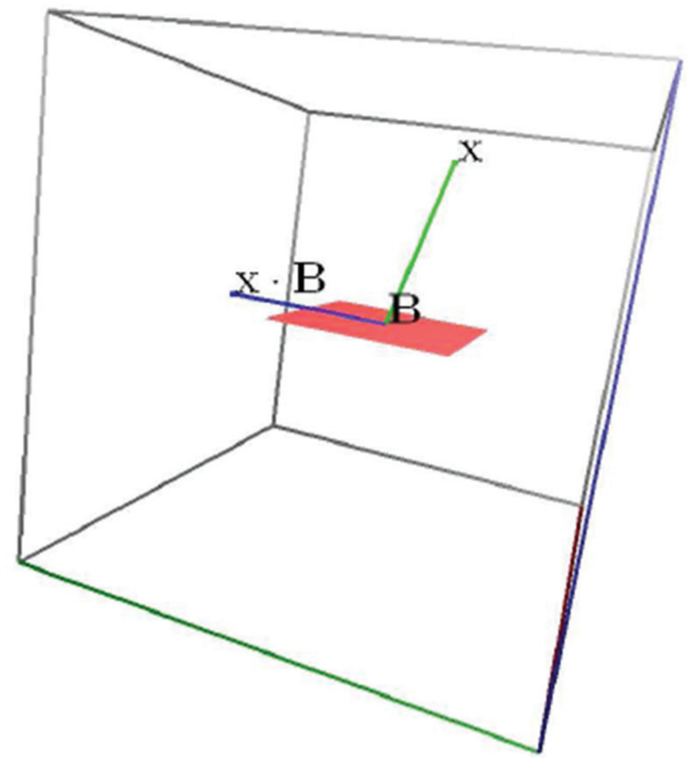
- Inner product not only defined for vectors!
- Example:

```
DefVarsE3();

:Red;
:B = e1 ^ e2;
?norm = B.B;

:Green;
:x = e1+e3;

:Blue;
// xiB is a vector in the B-plane
// perpendicular to x
:xiB = x.B;
```




- Note: - the resulting vector is perpendicular to x
-            - the inner product is grade decreasing

*InnerProductE3.clu*

# The general inner product

$$\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b}.$$

$x \quad e_1 \quad e_2$

$$x \cdot B = (e_1 + e_3) \cdot (e_1 \wedge e_2)$$

$$= (\underbrace{x \cdot e_1}_{1})\, e_2 - (\underbrace{x \cdot e_2}_{0})\, e_1$$

$$= e_2$$

$$x \cdot e_1$$

$$= (e_1 + e_3) \cdot e_1$$

$$= \underbrace{e_1 \cdot e_1}_{1} + \underbrace{e_3 \cdot e_1}_{0}$$

$$= 1$$

$$(e_1 + e_3) \cdot e_2$$

$$= (\underbrace{e_1 \cdot e_3}_{0}) + (\underbrace{e_3 \cdot e_2}_{0})$$

$$= 0$$

# The Geometric Product (Page 20)

$$uv = u \wedge v + u \cdot v$$

$u, v$ vectors

$$u^2 = uu = \underbrace{u \wedge u}_{0} + u \cdot u$$

$$= u \cdot u$$

Example:

$$e_1^2 = e_1 \cdot e_1 = 1$$

# The Geometric Product (Page 21)

$$(e_1 \wedge e_2)^2 = (e_1 \wedge e_2)(e_1 \wedge e_2)$$
$$= (e_1 e_2)(e_1 e_2)$$
$$= e_1 e_2 \underbrace{e_1 e_2}_{-e_2 e_1}$$
$$= -e_1 \underbrace{e_2 e_2}_{1} e_1$$
$$= -\underbrace{e_1 e_1}_{1}$$
$$= -1$$

$$e_1 e_2 = e_1 \wedge e_2 + \underbrace{e_1 \cdot e_2}_{0}$$

$$e_1 e_2 = e_1 \wedge e_2$$

gilt nur für ungleiche Indices:

$$e_1 e_1 = \underbrace{e_1 \wedge e_1}_{0} + \underbrace{e_1 \cdot e_1}_{1}$$
$$= 1$$

⚠ $e_1 \wedge e_2$ kann mit der imaginären Einheit $i$ identifiziert werden.