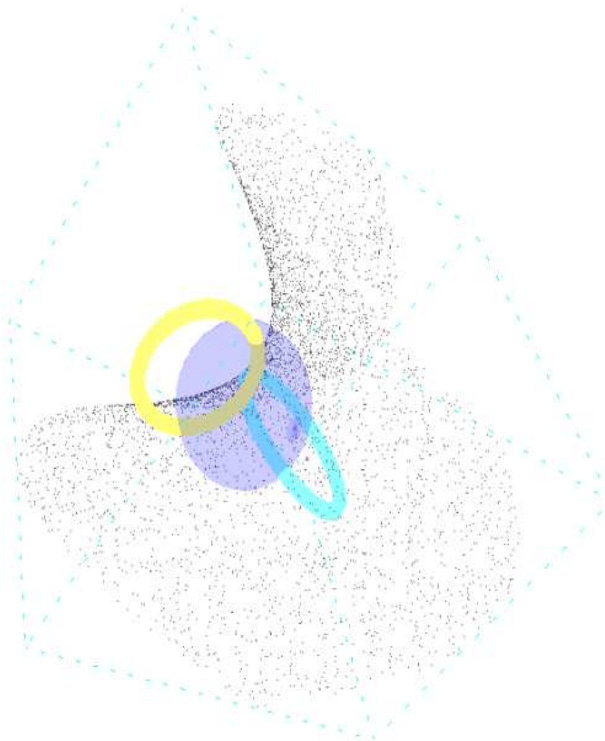
Geometric Algebra Computing
Analysis of point clouds
10.07.2019



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Literature

- Book „Foundations of Geometric Algebra Computing“, Dietmar Hildenbrand

- **Computers & Graphics 2005 :**
  volume 29, no. 5, october, 2005 :
  "Geometric Computing in Computer Graphics using Conformal Geometric Algebra" by Dietmar Hildenbrand

- **GRAPP 2008, Madeira :**
  "ANALYSIS OF POINT CLOUDS Using Conformal Geometric Algebra" by Dietmar Hildenbrand and Eckhard Hitzer.

- **VISAPP 2010:**
  „ESTIMATION OF CURVATURES IN POINT SETS BASED ON GEOMETRIC ALGEBRA“ by H. Seibert, D. Hildenbrand et. al.

Inner Product Calculations in 5D conformal GA

The geometric product of 2 basis vectors (revisited)

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$$\text{geometric algebra } G_{p,q}$$

$$with \quad n = p + q$$

$$\rightarrow$$

$$define$$

$$e_i e_j = \begin{cases} 1 & for \quad i = j \in \{1,\dots,p\} \\ -1 & for \quad i = j \in \{p+1,\dots,n\} \\ e_{ij} = e_i \wedge e_j = -e_j \wedge e_i & for \quad i \neq j \end{cases}$$

Note :

Conformal Geometric Algebra = $G_{4,1}$:

The two additional base vectors

The Conformal Geometric Algebra uses 2 additional base vectors (e_+, e_-) with the following properties.

$$e_+^2 = 1 \qquad e_-^2 = -1 \qquad e_+ \cdot e_- = 0 \qquad (3.1)$$

Another base (e_∞, e_o) can be defined with the following relations

$$e_o = \frac{1}{2}(e_- - e_+) \qquad e_\infty = e_- + e_+$$

The two additional base vectors

- … are null vectors

$$e_o^2 = e_\infty^2 = 0, \qquad e_\infty \cdot e_o = -1$$

$$e_- = e_o + \frac{1}{2}e_\infty \qquad e_+ = \frac{1}{2}e_\infty - e_o$$

The outer product $e_\infty \wedge e_o$ is often abbreviated by E.

The inner product between conformal vectors

$$P \cdot S = (\mathbf{p} + p_4 e_\infty + p_5 e_o) \cdot (\mathbf{s} + s_4 e_\infty + s_5 e_o)$$

$$= \mathbf{p} \cdot \mathbf{s} + s_4 \underbrace{\mathbf{p} \cdot e_\infty}_{0} + s_5 \underbrace{\mathbf{p} \cdot e_o}_{0}$$

$$+ p_4 \underbrace{e_\infty \cdot \mathbf{s}}_{0} + p_4 s_4 \underbrace{e_\infty^2}_{0} + p_4 s_5 \underbrace{e_\infty \cdot e_o}_{-1}$$

$$+ p_5 \underbrace{e_o \cdot \mathbf{s}}_{0} + p_5 s_4 \underbrace{e_o \cdot e_\infty}_{-1} + p_5 s_5 \underbrace{e_o^2}_{0}$$

It results in

$$P \cdot S = \mathbf{p} \cdot \mathbf{s} - p_5 s_4 - p_4 s_5 \qquad (3.6)$$

or

$$P \cdot S = p_1 s_1 + p_2 s_2 + p_3 s_3 - p_5 s_4 - p_4 s_5$$

The distance between points

In the case of P and S being points we get

$$p_4 = \frac{1}{2}\mathbf{p}^2, p_5 = 1$$

$$s_4 = \frac{1}{2}\mathbf{s}^2, s_5 = 1$$

The inner product of these points is according to equation 3.6

$$P \cdot S = \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}\mathbf{s}^2 - \frac{1}{2}\mathbf{p}^2$$

$$= -\frac{1}{2}(\mathbf{s} - \mathbf{p})^2$$

We recognize that the square of the Euclidean distance of the inhomogenous points corresponds to the inner product of the homogenous points multiplied by -2.

$$(\mathbf{s} - \mathbf{p})^2 = -2(P \cdot S)$$

Distance between point and plane

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For a vector P representing a point we get

$$p_4 = \frac{1}{2}\mathbf{p}^2, p_5 = 1$$

For a vector S representing a plane with normal vector $\mathbf{n}$ and distance d we get

$$\mathbf{s} = \mathbf{n}, s_4 = d, s_5 = 0$$

The inner product of point and plane is according to equation 3.6

$$P \cdot S = \mathbf{p} \cdot \mathbf{n} - d$$

representing the Euclidean distance of a point and a plane.

Point inside or outside of a sphere?

For a vector S representing a sphere we get

$$s_4 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2 - r^2), \qquad s_5 = 1$$

The inner product of point and sphere is according to equation 3.6

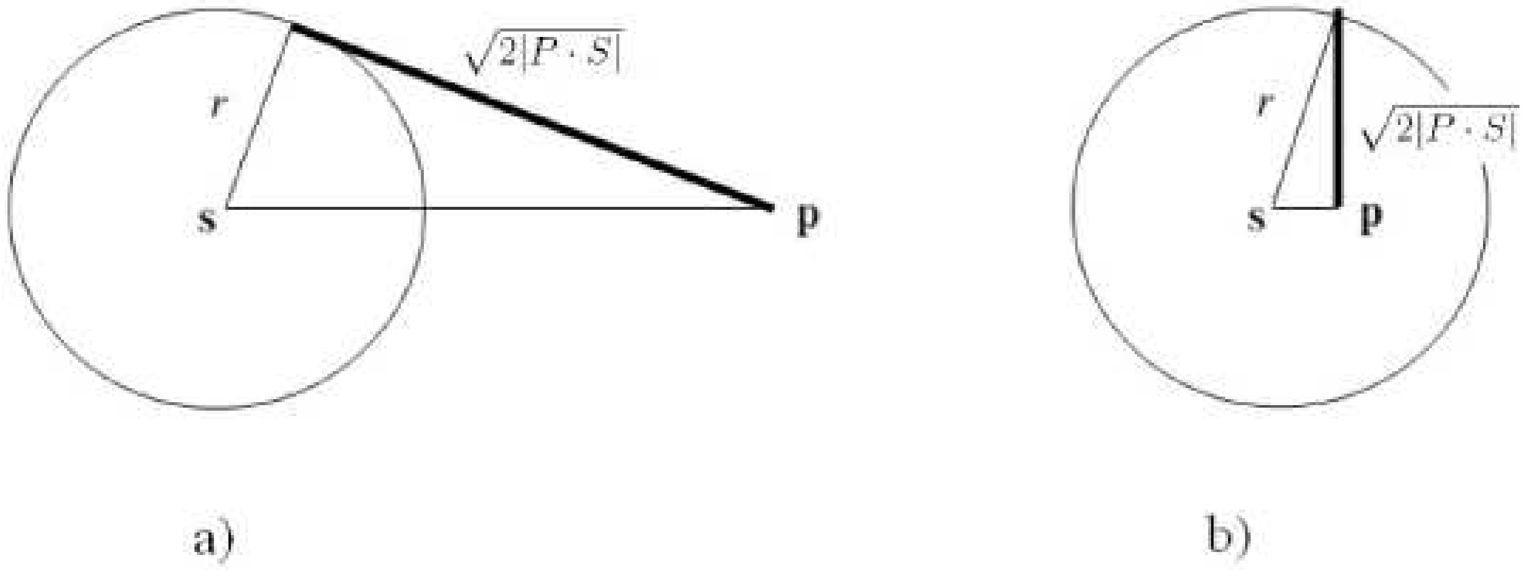
$$\begin{aligned} P \cdot S &= \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}(\mathbf{s}^2 - r^2) - \frac{1}{2}\mathbf{p}^2 \\ &= \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}\mathbf{s}^2 + \frac{1}{2}r^2 - \frac{1}{2}\mathbf{p}^2 \\ &= \frac{1}{2}r^2 - \frac{1}{2}(\mathbf{s}^2 - 2\mathbf{p} \cdot \mathbf{s} - \mathbf{p}^2) \\ &= \frac{1}{2}r^2 - \frac{1}{2}(\mathbf{s} - \mathbf{p})^2 \end{aligned}$$

We get

$$2(P \cdot S) = r^2 - (\mathbf{s} - \mathbf{p})^2$$

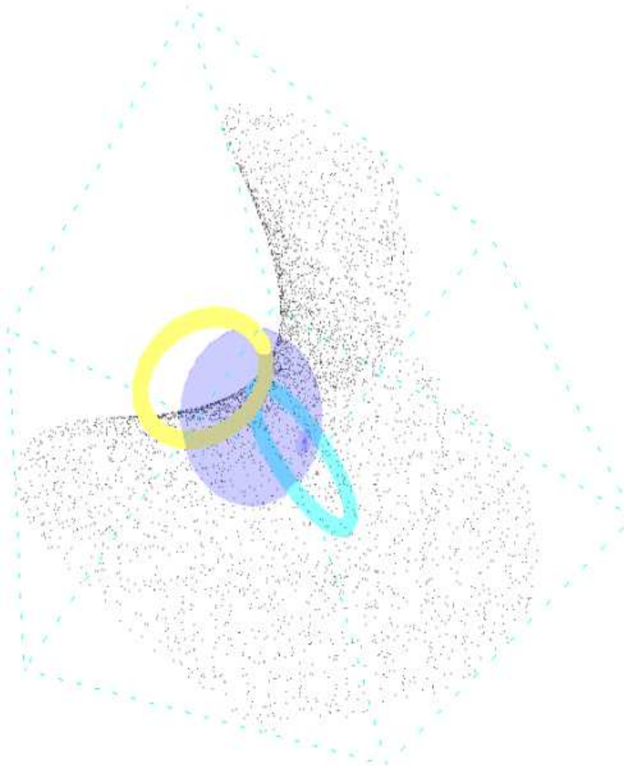
$P \cdot S > 0 :$ $\mathbf{p}$ is inside of the sphere
$P \cdot S = 0 :$ $\mathbf{p}$ is on the sphere
$P \cdot S < 0 :$ $\mathbf{p}$ is outside of the sphere

Distance measure: Inner product of point and sphere



Analysis of point clouds

- normals
- curvatures
- …



Analysis of point clouds

- normals
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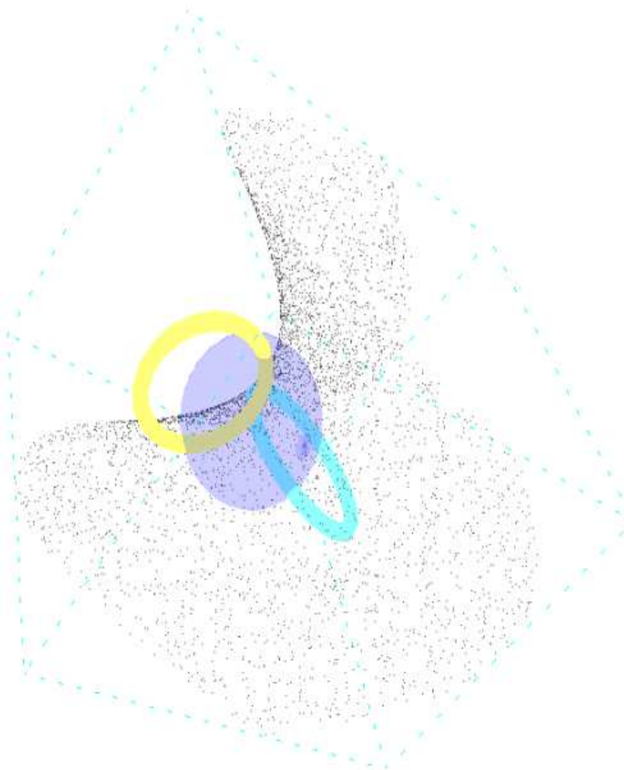


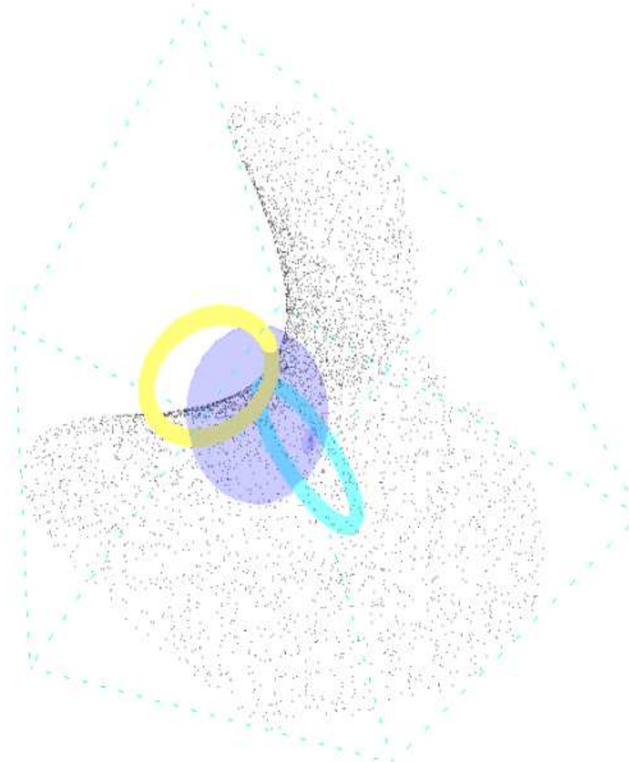
What are the most interesting local fittings of geometric objects?

Fitting of geometric objects into point clouds

- Plane fitting
  - Normal vector

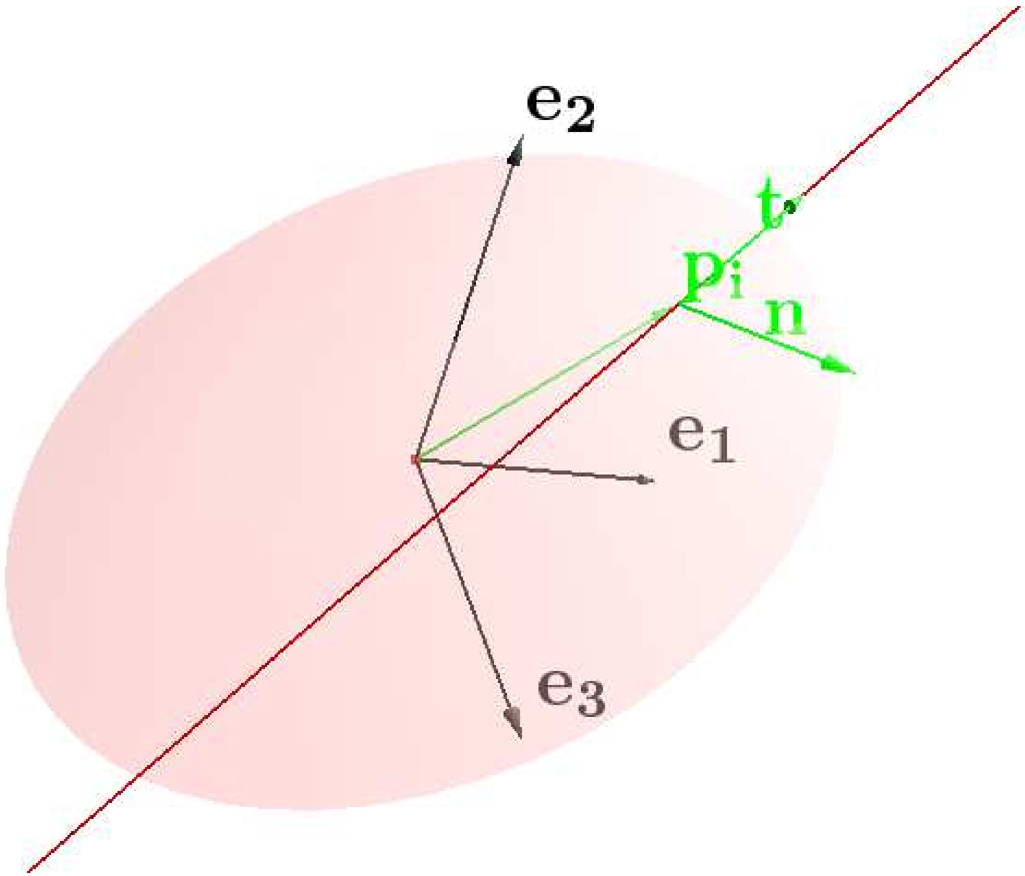
- Sphere fitting
  - curvature

  Note.: osculating circle in tangent direction



Curvature =0 at point p_i

- Line $\kappa = 0$
- Circle $\kappa > 0$
- Point $\kappa \rightarrow \infty$



Curvature >0 at point p_i

- Line $\kappa = 0$
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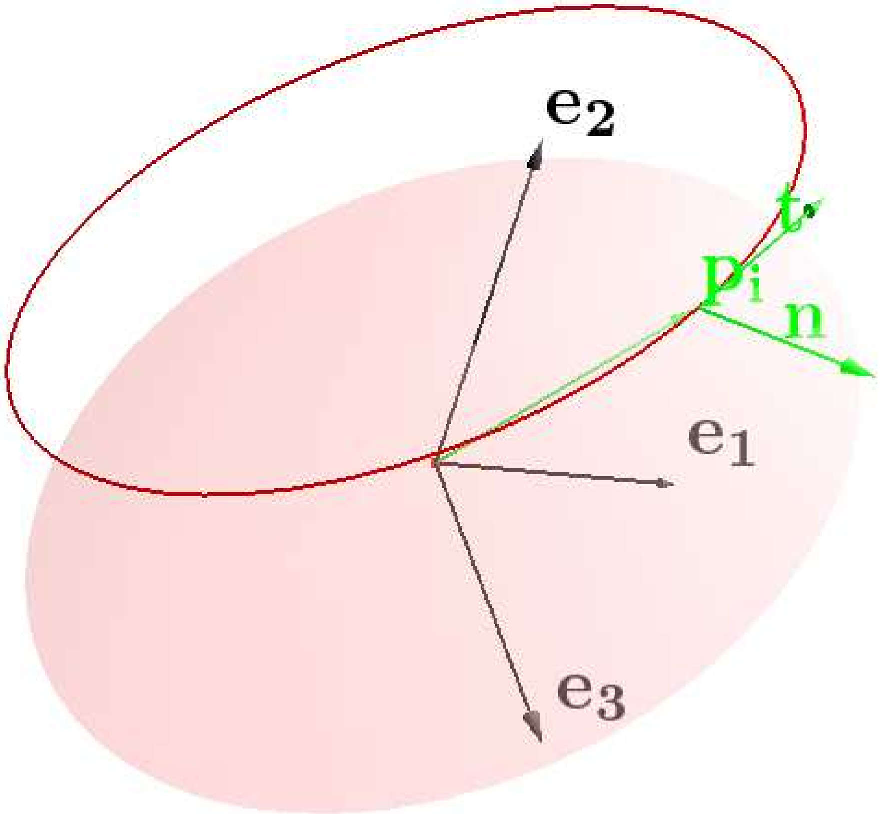
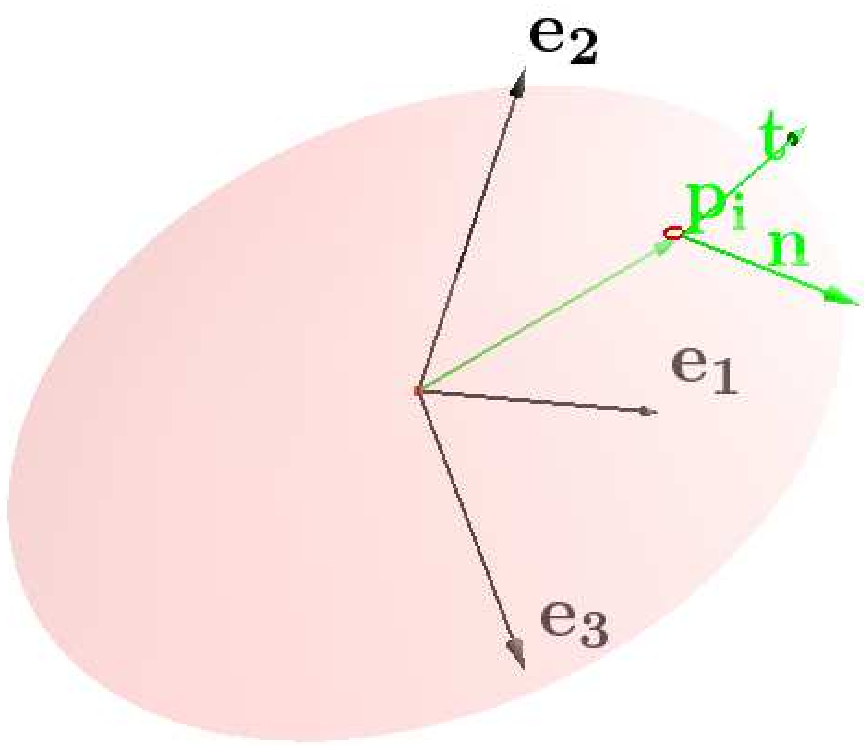
- Point $\kappa \to \infty$



Infinite curvature at point p_i

- Line $\kappa = 0$
- Circle $\kappa > 0$
- Point $\kappa \to \infty$



Overview

- Conventional fitting of spheres
- Fitting of spheres in GA
- The role of infinity
- Planes as a limit of spheres
- Fitting of spheres or planes in GA
- Fitting of osculating circles in point clouds

Conventional fitting of spheres

Given a set of points $\{(x_i, y_i, z_i)\}_{i=1}^m$, $m \geq 4$, fit them with a sphere $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$, where (a, b, c) is the sphere center and r is the sphere radius. An assumption of this algorithm is that not all the points are coplanar. The error function to be minimized is

$$E(a, b, c, r) = \sum_{i=1}^{m} (L_i - r)^2$$

where $L_i = \sqrt{(x_i - a)^2 + (y_i - b)^2 + (z_i - c)^2}$.

[Eberly, 2007]

- Note: sphere isn´t one algebraic expression

Conventional fitting of spheres

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Take the partial derivative with respect to r to obtain

$$\frac{\partial E}{\partial r} = -2 \sum_{i=1}^{m} (L_i - r)$$

Setting equal to zero yields

$$r = \frac{1}{m} \sum_{i=1}^{m} L_i$$

Take the partial derivative with respect to a to obtain

$$\frac{\partial E}{\partial a} = 2 \sum_{i=1}^{m} (L_i - r) \frac{\partial L_i}{\partial a} = -2 \sum_{i=1}^{m} \left((x_i - a) + r \frac{\partial L_i}{\partial a} \right)$$

take the partial derivative with respect to b to obtain

$$\frac{\partial E}{\partial b} = 2 \sum_{i=1}^{m} (L_i - r) \frac{\partial L_i}{\partial b} = -2 \sum_{i=1}^{m} \left((y_i - b) + r \frac{\partial L_i}{\partial b} \right)$$

Conventional fitting of spheres

Setting these three derivatives equal to zero yields

$$a = \frac{1}{m}\sum_{i=1}^{m} x_i + r\frac{1}{m}\sum_{i=1}^{m}\frac{\partial L_i}{\partial a}$$

and

$$b = \frac{1}{m}\sum_{i=1}^{m} y_i + r\frac{1}{m}\sum_{i=1}^{m}\frac{\partial L_i}{\partial b}$$

and

$$c = \frac{1}{m}\sum_{i=1}^{m} z_i + r\frac{1}{m}\sum_{i=1}^{m}\frac{\partial L_i}{\partial c}$$

Conventional fitting of spheres

Replacing r by its equivalent from $\partial E/\partial r = 0$ and using $\partial L_i/\partial a = (a - x_i)/L_i$, $\partial L_i/\partial b = (b - y_i)/L_i$, and $\partial L_i/\partial c = (c - z_i)/L_i$ leads to three nonlinear equations in a, b, and c:

$$a = \bar{x} + \bar{L}\bar{L}_a =: F(a, b, c)$$

$$b = \bar{y} + \bar{L}\bar{L}_b =: G(a, b, c)$$

$$c = \bar{z} + \bar{L}\bar{L}_c =: H(a, b, c)$$

where

$$\bar{x} = \frac{1}{m}\sum_{i=1}^{m} x_i$$

$$\bar{y} = \frac{1}{m}\sum_{i=1}^{m} y_i$$

$$\bar{z} = \frac{1}{m}\sum_{i=1}^{m} z_i$$

$$\bar{L} = \frac{1}{m}\sum_{i=1}^{m} L_i$$

Conventional fitting of spheres

$$\bar{L}_a = \frac{1}{m} \sum_{i=1}^{m} \frac{a - x_i}{L_i}$$

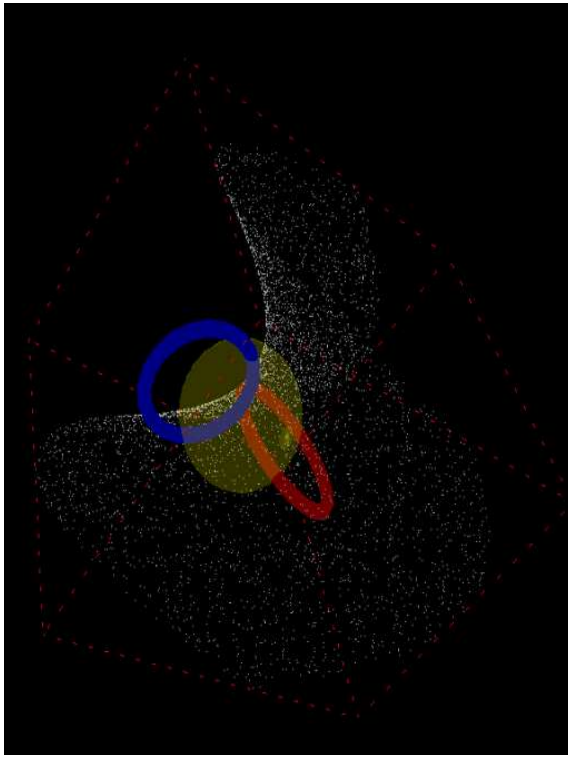
$$\bar{L}_b = \frac{1}{m} \sum_{i=1}^{m} \frac{b - y_i}{L_i}$$

$$\bar{L}_c = \frac{1}{m} \sum_{i=1}^{m} \frac{c - z_i}{L_i}$$

Fixed-point iteration can be applied to solving these equations: $a_0 = \bar{x}$, $b_0 = \bar{y}$, $c_0 = \bar{z}$, and $a_{i+1} = F(a_i, b_i, c_i)$, $b_{i+1} = G(a_i, b_i, c_i)$, and $c_{i+1} = H(a_i, b_i, c_i)$ for $i \geq 0$.

Benefits of geometric algebra

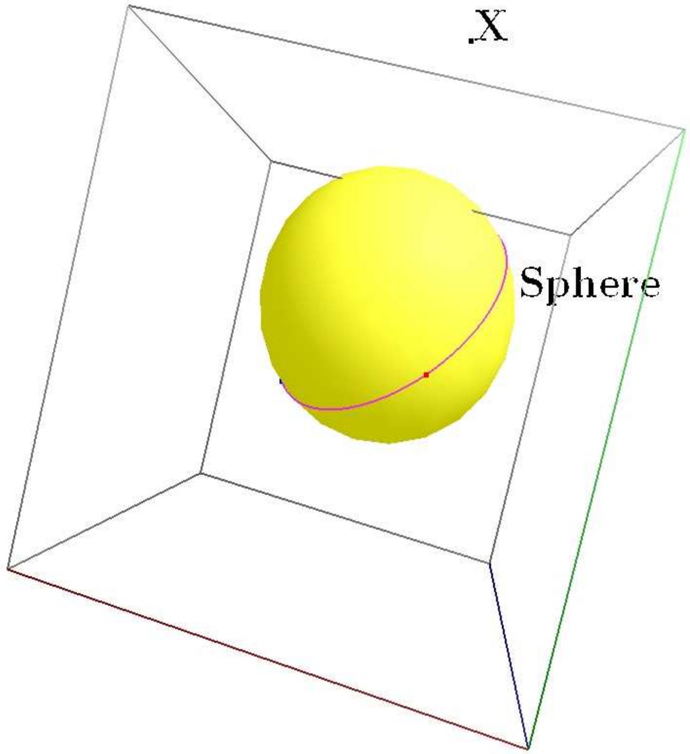
- Easy computations with algebraic objects describing spheres, planes and circles …



Fitting a sphere to 3D points

- Distance measure for the fitting?
  - point P
  - sphere S

$P \cdot S > 0$: **p** is inside of the sphere
$P \cdot S = 0$: **p** is on the sphere
$P \cdot S < 0$: **p** is outside of the sphere



Inner product of point and sphere revisited

For a vector S representing a sphere we get

$$s_4 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2 - r^2), \qquad s_5 = 1$$

The inner product of point and sphere is according to equation 3.6

$$\begin{aligned} P \cdot S &= \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}(\mathbf{s}^2 - r^2) - \frac{1}{2}\mathbf{p}^2 \\ &= \mathbf{p} \cdot \mathbf{s} - \frac{1}{2}\mathbf{s}^2 + \frac{1}{2}r^2 - \frac{1}{2}\mathbf{p}^2 \\ &= \frac{1}{2}r^2 - \frac{1}{2}(\mathbf{s}^2 - 2\mathbf{p} \cdot \mathbf{s} - \mathbf{p}^2) \\ &= \frac{1}{2}r^2 - \frac{1}{2}(\mathbf{s} - \mathbf{p})^2 \end{aligned}$$

We get

$$2(P \cdot S) = r^2 - (\mathbf{s} - \mathbf{p})^2$$

$P \cdot S > 0$: $\mathbf{p}$ is inside of the sphere
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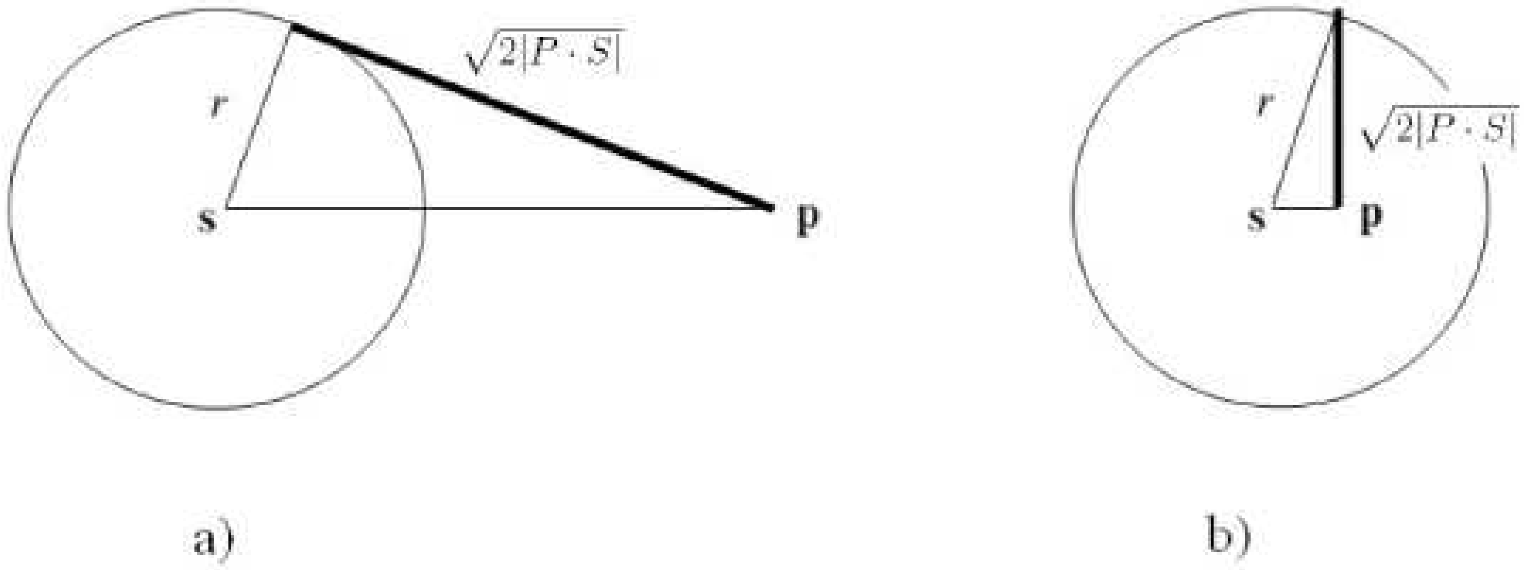
Distance measure: Inner product of point and sphere



Fitting a sphere to 3D points

a point set $\mathbf{p_i} \in \mathbb{R}^3,\ i \in \{1, ..., n\}$ will be approximated with the help of a sphere. The inhomogenous points $\mathbf{p_i}$ are represented as

$$P_i = \mathbf{p_i} + \frac{1}{2}\mathbf{p_i}^2 e_\infty + e_0 \tag{6.16}$$

and the sphere S with inhomogenous center point $\mathbf{s}$ and radius r is represented as

$$S = \mathbf{s} + s_4 e_\infty + e_0 \tag{6.17}$$

with

$$s_4 = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2 - r_2^2)$$

[Dissertation Hildenbrand]

- Correction: r^2

Fitting a sphere to 3D points

The inner product between a point P_i and the sphere S is defined by

$$P_i \cdot S = (\mathbf{p_i} + \frac{1}{2}\mathbf{p_i}^2 e_\infty + e_0) \cdot (\mathbf{s} + s_4 e_\infty + e_0)$$

this results in

$$P_i \cdot S = \mathbf{p_i} \cdot \mathbf{s} - \frac{1}{2}\mathbf{p_i}^2 - s_4$$

or

$$P_i \cdot S = w_{i,1}s_1 + w_{i,2}s_2 + w_{i,3}s_3 + w_{i,4}s_4 + w_{i,5} = \sum_{j=1}^{4}(w_{i,j}s_j) + w_{i,5}$$

with

$$w_{i,k} = \begin{cases} p_{i,k} & : \quad k \in \{1,2,3\} \\ -1 & : \quad k = 4 \\ -\frac{1}{2}\mathbf{p_i}^2 & : \quad k = 5 \end{cases}$$

Fitting a sphere to 3D points

$$min \sum_{i=1}^{n} (P_i \cdot S)^2$$

In order to obtain the minimum we have the following 4 necessary conditions

$$\forall k \in \{1..4\} : \frac{\partial(\sum_{i=1}^{n} (P_i \cdot S)^2)}{\partial s_k} = \sum_{i=1}^{n} \frac{\partial(P_i \cdot S)^2}{\partial s_k} = 0$$

With the help of

$$\frac{\partial(P_i \cdot S)^2}{\partial s_k} = 2(P_i \cdot S) \cdot \frac{\partial(P_i \cdot S)}{\partial s_k}$$

and

$$\frac{\partial(P_i \cdot S)}{\partial s_k} = \frac{\partial(\sum_{j=1}^{4}(w_{i,j}s_j) + w_{i,5})}{\partial s_k} = w_{i,k}$$

we obtain

$$\forall k \in \{1..4\} : \frac{\partial(P_i \cdot S)^2}{\partial s_k} = 2\sum_{i=1}^{n}(\sum_{j=1}^{4}(w_{i,j}s_jw_{i,k}) + w_{i,5}w_{i,k}) = 0$$

- Correction: Missing sum sign

Fitting a sphere to 3D points

The result of the least squares approach is as follows :

$$\begin{pmatrix} \sum_{i=1}^n p_{i,1}p_{i,1} & \sum_{i=1}^n p_{i,2}p_{i,1} & \sum_{i=1}^n p_{i,3}p_{i,1} & -\sum_{i=1}^n p_{i,1} \\ \sum_{i=1}^n p_{i,1}p_{i,2} & \sum_{i=1}^n p_{i,2}p_{i,2} & \sum_{i=1}^n p_{i,3}p_{i,2} & -\sum_{i=1}^n p_{i,2} \\ \sum_{i=1}^n p_{i,1}p_{i,3} & \sum_{i=1}^n p_{i,2}p_{i,3} & \sum_{i=1}^n p_{i,3}p_{i,3} & -\sum_{i=1}^n p_{i,3} \\ -\sum_{i=1}^n p_{i,1} & -\sum_{i=1}^n p_{i,2} & -\sum_{i=1}^n p_{i,3} & \sum_{i=1}^n 1 \end{pmatrix} \cdot s = \begin{pmatrix} \frac{1}{2}\sum_{i=1}^n \mathbf{p_i}^2 p_{i,1} \\ \frac{1}{2}\sum_{i=1}^n \mathbf{p_i}^2 p_{i,2} \\ \frac{1}{2}\sum_{i=1}^n \mathbf{p_i}^2 p_{i,3} \\ -\frac{1}{2}\sum_{i=1}^n \mathbf{p_i}^2 \end{pmatrix} \quad (6.22)$$

with $p_{i,1}, p_{i,2}, p_{i,3}$ as inhomogenous coordinates of the points $\mathbf{p}_i$. The result $s = (s_1, s_2, s_3, s_4)$ represents the center point of the sphere (s_1, s_2, s_3) and its radius in terms of $r^2 = s_1^2 + s_2^2 + s_3^2 - 2s_4$

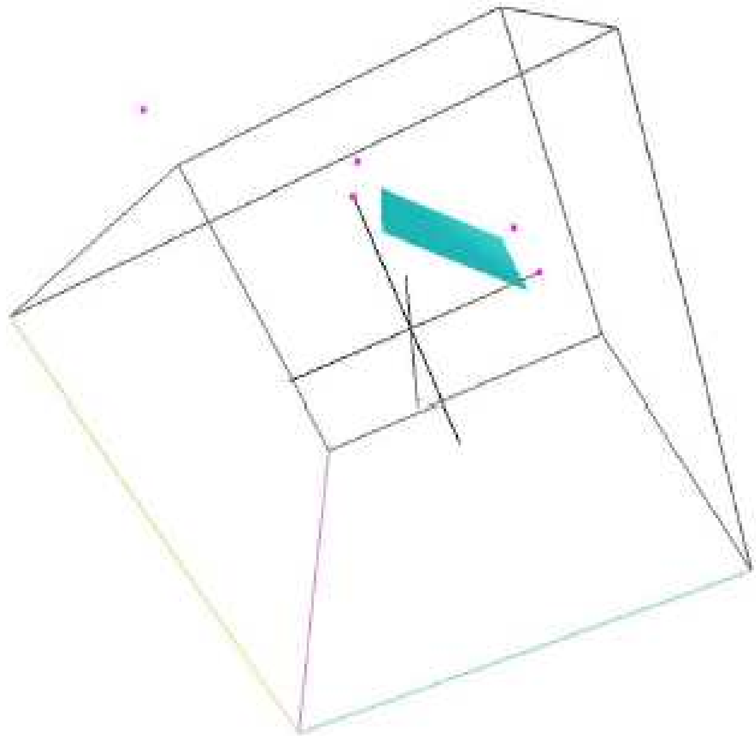
Fitting of sphere *or* plane into a point cloud

In CGA : 5D vector with shortest distance measure to the points ?

- Planes and spheres are vectors
- Inner product as a distance measure
- Least squares approach

$$min \sum_{i=1}^{n} (X_i \cdot S)^2$$

- Result: eigen vectors of 5x5 matrix



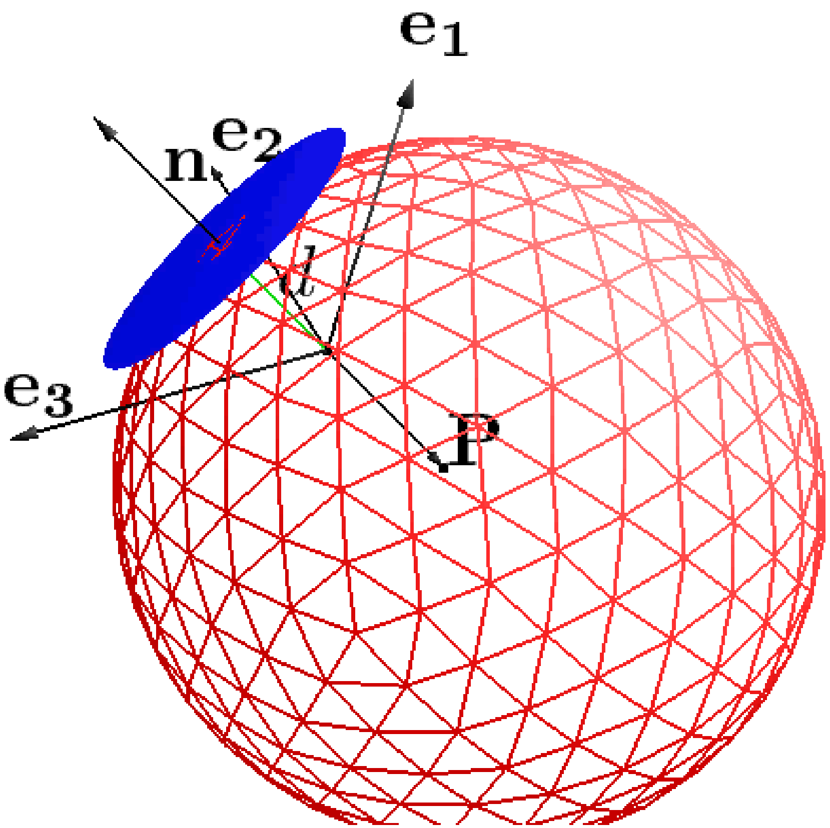
Plane as a specific sphere

- sphere

$$S = P - \frac{1}{2}r^2 e_\infty$$

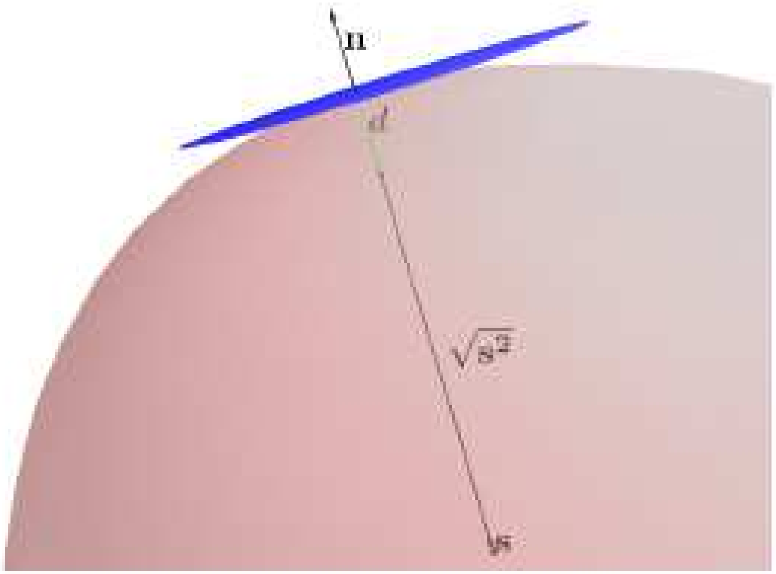
- plane

$$\pi = \mathbf{n} + d e_\infty$$



Plane as a limit of spheres

- What happens with infinitely increasing radius?
- at first: What is the role of infinity?



Origin sphere with infinite radius?

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this origin sphere ($p = e_0$) is represented as

$$S = -\frac{1}{2}r^2 e_\infty + e_0 \tag{4.1}$$

Another homogenous representation of this origin sphere is for instance its product with the scalar $-\frac{2}{r^2}$

$$S' = -\frac{2}{r^2}S = e_\infty - \frac{2}{r^2}e_0 \tag{4.2}$$

Based on this formula and on the fact that S and S' are representing the same sphere we can easily see that an origin sphere with infinite radius is represented by e_∞

$$\lim_{r\to\infty} S' = e_\infty$$

It can be shown that this is true not only for an origin sphere but also for a sphere with an arbitrary center point.

- Exercise?

Point at infinity?

Let us now assume an arbitrary Euclidean point $\mathbf{x}$ (not equal to the origin) represented by the conformal vector P

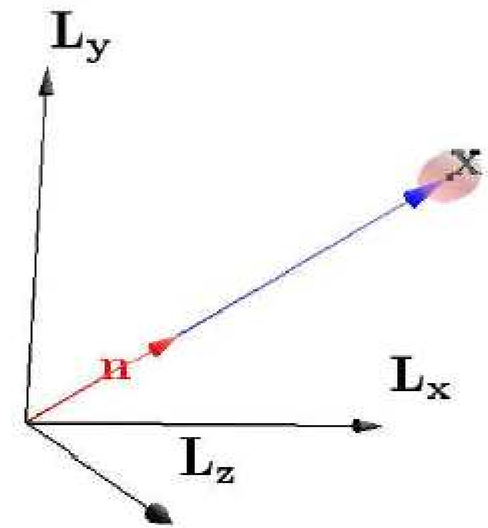
$$P = \mathbf{x} + \frac{1}{2}\mathbf{x}^2 e_\infty + e_0$$

with the Euclidean normal vector $\mathbf{n}$ in the direction of $\mathbf{x}$

$$\mathbf{x} = t\mathbf{n},\ t > 0,\ \mathbf{n}^2 = 1$$

Another homogenous representation of this point is for instance its product with the scalar $\frac{2}{\mathbf{x}^2}$

$$P' = \frac{2}{\mathbf{x}^2}(\mathbf{x} + \frac{1}{2}\mathbf{x}^2 e_\infty + e_0)$$



Point at infinity?

$$P' = \frac{2}{\mathbf{x}^2}\mathbf{x} + e_\infty + \frac{2}{\mathbf{x}^2}e_0$$

We use that form in order to compute the limit $\lim_{t\to\infty} P'$ for increasing $\mathbf{x}$. Since $\mathbf{x} = t\mathbf{n}$ we get

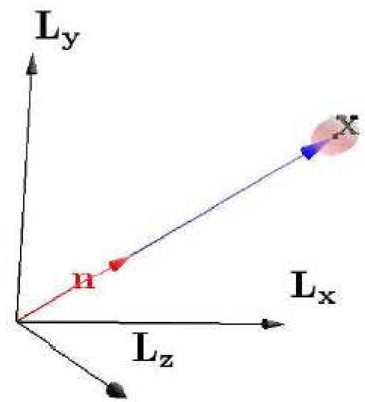
$$P' = \frac{2}{t^2\mathbf{n}^2}t\mathbf{n} + e_\infty + \frac{2}{t^2\mathbf{n}^2}e_0$$

and since $\mathbf{n}^2 = 1$

$$P' = \frac{2}{t}\mathbf{n} + e_\infty + \frac{2}{t^2}e_0$$

Based on this formula and on the fact that P and P' are representing the same Euclidean point we can easily see that the point at infinity for each direction vector $\mathbf{n}$ is represented by e_∞

$$\lim_{t\to\infty} P' = e_\infty$$



Plane at infinity?

Let us consider a plane with an arbitrary distance $d \neq 0$ to the origin. According to equation 2.26 this plane is represented as

$$\pi = \mathbf{n} + de_\infty \tag{4.3}$$

with the 3D normal vector $\mathbf{n}$. Another homogenous representation of this plane is for instance its product with the scalar $\frac{1}{d}$

$$\pi' = \frac{1}{d}\mathbf{n} + e_\infty \tag{4.4}$$

Based on this formula and on the fact that π and π' are representing the same plane we can easily see that a plane with infinite distance to the origin is represented by e_∞

$$\lim_{d \to \infty} \pi' = e_\infty$$

Plane as a limit of spheres

a sphere S

$$S = \mathbf{s} + \frac{1}{2}(\mathbf{s}^2 - r^2)e_\infty + e_0 \qquad (4.5)$$

with Euclidean center point $\mathbf{s}$ and radius r degenerates to a plane as the result of a limit process.
Depending on whether the origin lies inside or outside of the sphere the minimum distance from the origin to the sphere is

$$d = r \pm \sqrt{\mathbf{s}^2}$$

If the origin lies inside of the sphere

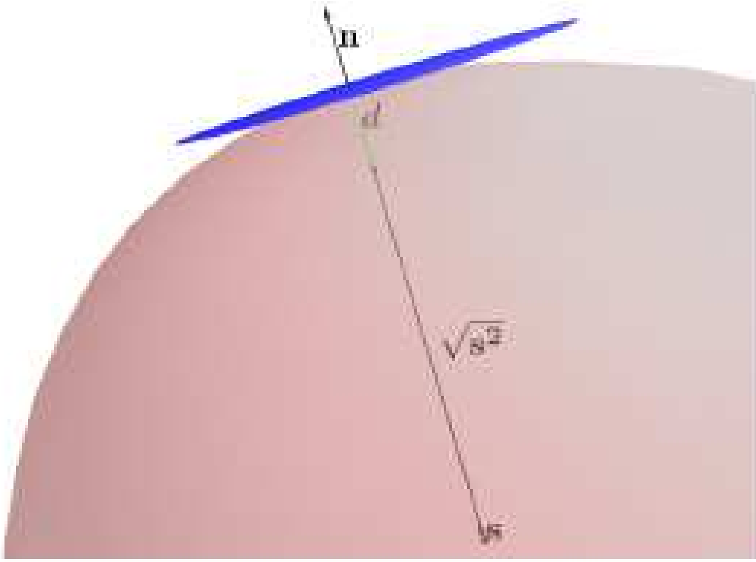
$$r = \sqrt{\mathbf{s}^2} + d$$

or

$$r^2 = \mathbf{s}^2 + 2d\sqrt{\mathbf{s}^2} + d^2$$

and the sphere can be written as

$$S = \mathbf{s} + \frac{1}{2}(\mathbf{s}^2 - \mathbf{s}^2 - 2d\sqrt{\mathbf{s}^2} - d^2)e_\infty + e_0$$



Plane as a limit of spheres

or

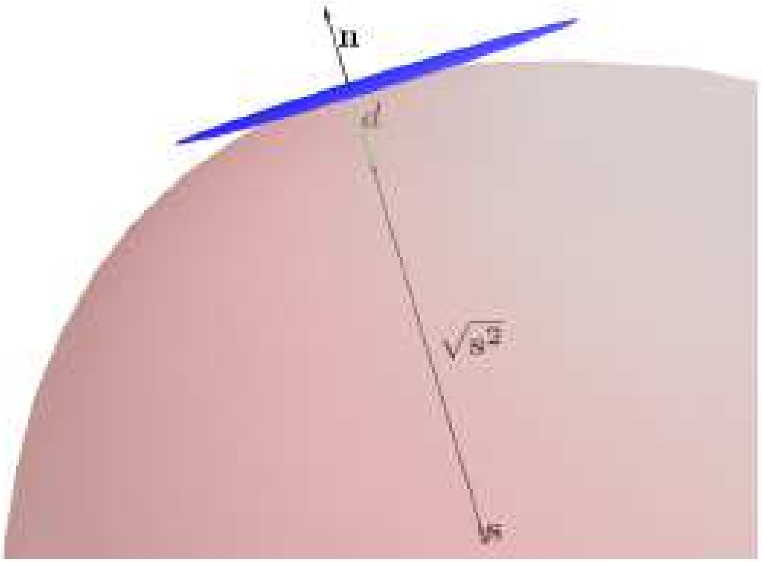
$$S = \mathbf{s} + \frac{1}{2}(-2d\sqrt{\mathbf{s}^2} - d^2)e_\infty + e_0$$

With $\mathbf{s} = -t\mathbf{n}$ and $\mathbf{s}^2 = t^2\mathbf{n}^2$ we get

$$S = -t\mathbf{n} - \frac{1}{2}(2td\sqrt{\mathbf{n}^2} + d^2)e_\infty + e_0$$

or

$$-\frac{S}{t} = \mathbf{n} + \frac{1}{2}(2d\sqrt{\mathbf{n}^2} + \frac{d^2}{t})e_\infty - \frac{e_0}{t}$$



$$\lim_{t\to\infty} -\frac{S}{t} = \mathbf{n} + \lim_{t\to\infty}\frac{1}{2}(2d + \frac{d^2}{t})e_\infty - \lim_{t\to\infty}\frac{e_0}{t}$$

$$\lim_{t\to\infty} -\frac{S}{t} = \mathbf{n} + de_\infty$$

Vectors in GA

$$S = s_1 e_1 + s_2 e_2 + s_3 e_3 + s_4 e_\infty + s_5 e_0$$

sphere

$$S = \mathbf{s} + s_4 e_\infty + e_0$$

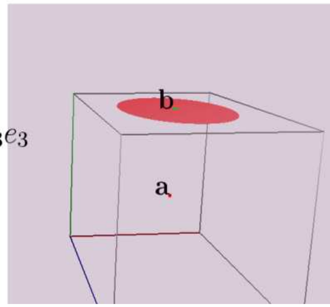
$$s_4 = \frac{1}{2}(\mathbf{s}^2 - r^2) = \frac{1}{2}(s_1^2 + s_2^2 + s_3^2 - r^2)$$

plane

$$\pi = \mathbf{n} + d\mathrm{e}_\infty$$

with

- the normal vector $\mathbf{n}$
  $= (n_1, n_2, n_3)$
  $= n_1 e_1 + n_2 e_2 + n_3 e_3$
- and distance d
  to the origin



Fitting of sphere *or* plane into a point cloud

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A distance measure between a point P_i and the sphere/plane S can be defined in Conformal Geometric Algebra with the help of their inner product

$$P_i \cdot S = (\mathbf{p_i} + \frac{1}{2}\mathbf{p_i}^2 e_\infty + e_0) \cdot (\mathbf{s} + s_4 e_\infty + s_5 e_0) \tag{5.25}$$

According to equation (5.10) this results in

$$P_i \cdot S = \mathbf{p_i} \cdot \mathbf{s} - s_4 - \frac{1}{2} s_5 \mathbf{p_i}^2$$

or

$$P_i \cdot S = \sum_{j=1}^{5} w_{i,j} s_j \tag{5.26}$$

with

$$w_{i,k} = \begin{cases} p_{i,k} & : \quad k \in \{1, 2, 3\} \\ -1 & : \quad k = 4 \\ -\frac{1}{2}\mathbf{p_i}^2 & : \quad k = 5 \end{cases}$$

Fitting of sphere *or* plane into a point cloud

- Least squares approach with constraint |s|=1 (Lagrange)

$$min \sum_{i=1}^{n} (X_i \cdot S)^2$$

- in bilinear form $min(s^T Bs)$ with

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} & b_{1,5} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} & b_{2,5} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} & b_{3,5} \\ b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & b_{4,5} \\ b_{5,1} & b_{5,2} & b_{5,3} & b_{5,4} & b_{5,5} \end{pmatrix}$$

$$b_{j,k} = \sum_{i=1}^{n} w_{i,j} w_{i,k}.$$

- Introduce L

$$L = s^T Bs - 0 = s^T Bs - \lambda(s^T s - 1),$$

$$s^T s = 1,$$

$$B^T = B$$

- Necessary condition

$$0 = \nabla L = 2 \cdot (Bs - \lambda s) = 0$$

- Eigen vector of B with smallest eigen value

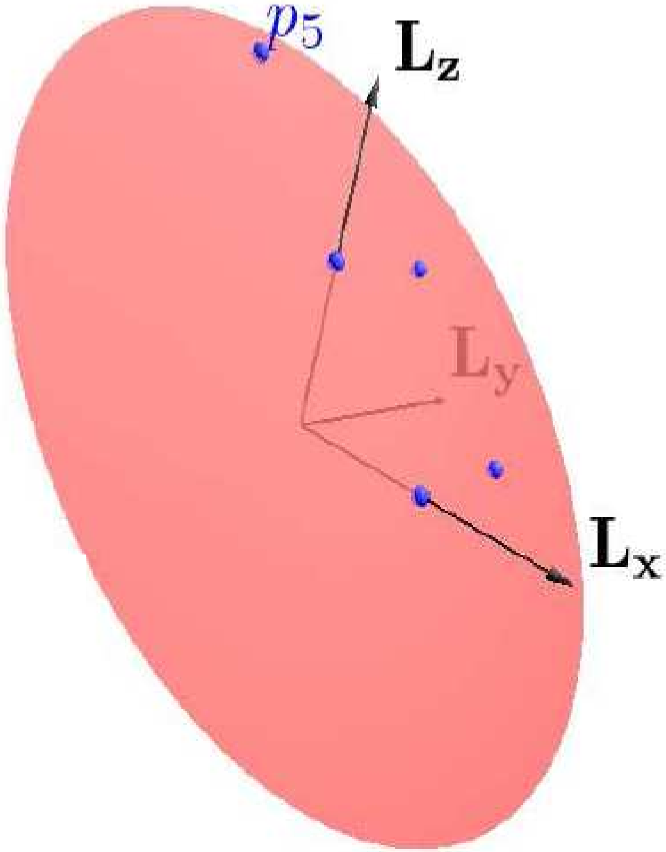
$$\rightarrow Bs = \lambda s$$

[„Matrix Analysis“, Horn/Johnson]

Results

Point	x	y	z
$\mathbf{p_1}$	1	0	0
$\mathbf{p_2}$	1	1	0
$\mathbf{p_3}$	0	0	1
$\mathbf{p_4}$	0	1	1
$\mathbf{p_5}$	-1	0	2

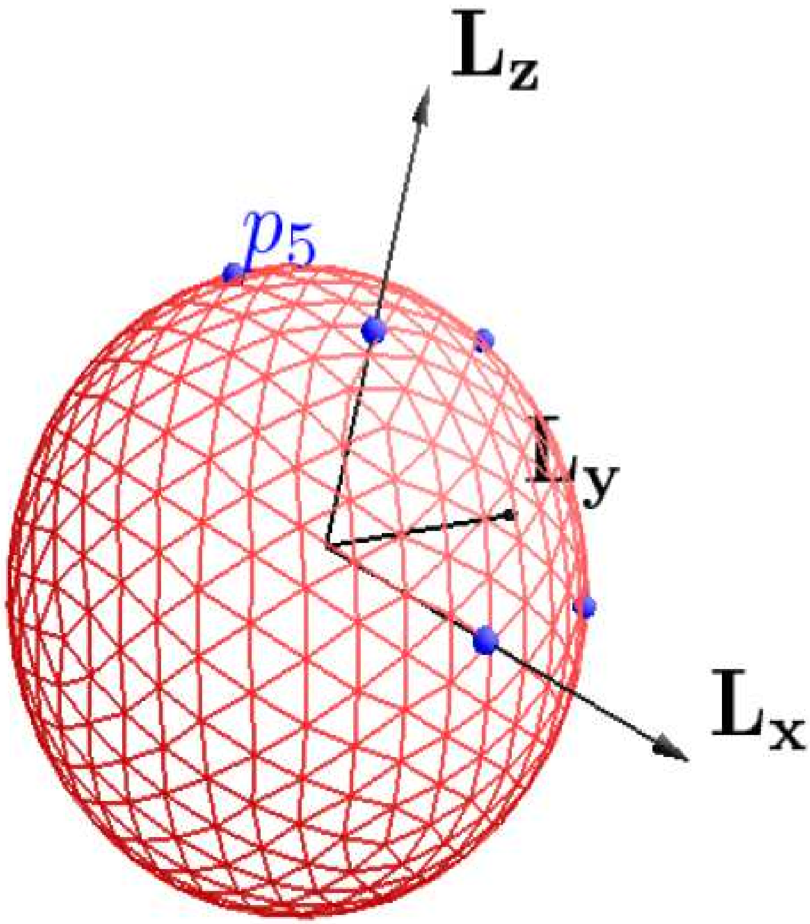
$$S = 0.57735 * e1 + 0.57735 * e3 + 0.57735 * e_\infty$$



Results

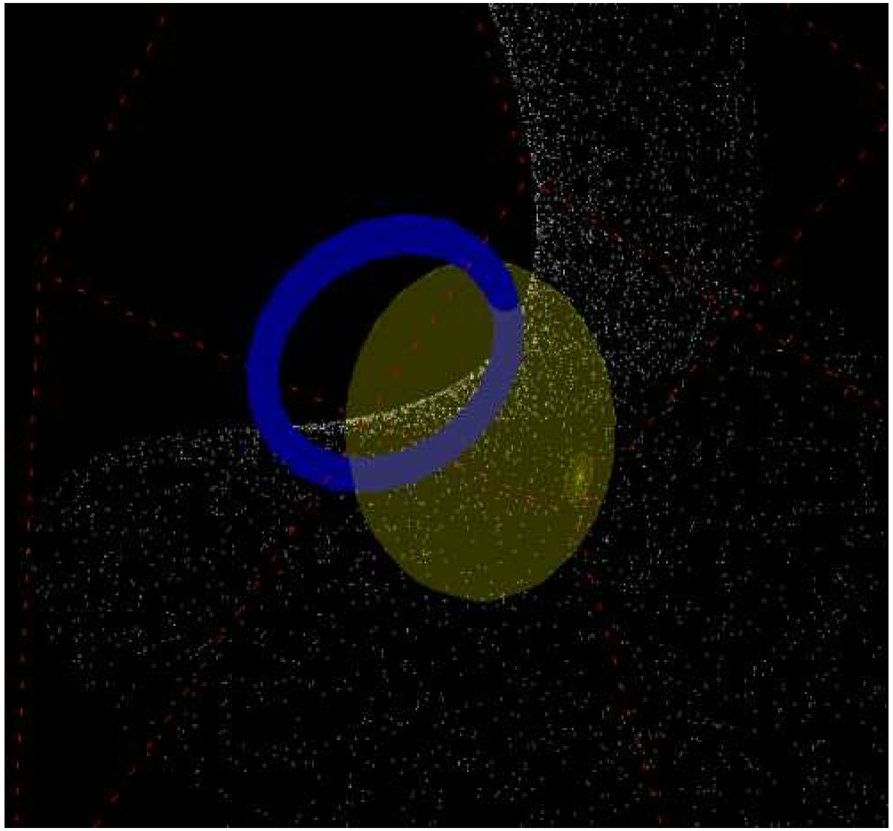
Point	x	y	z
$\mathbf{p}_1$	1	0	0
$\mathbf{p}_2$	1	1	0
$\mathbf{p}_3$	0	0	1
$\mathbf{p}_4$	0	1	1
$\mathbf{p}_5$	-1	0	1

$$S = -0.5e_1 + 0.5e_2 - 0.5e_3 - e_\infty + e_0$$



Curvature estimation for point clouds

- 2D:
  - Osculating circle
    -> curvature = inverse radius
- 3D
  - Same for the osculating circle in tangent direction

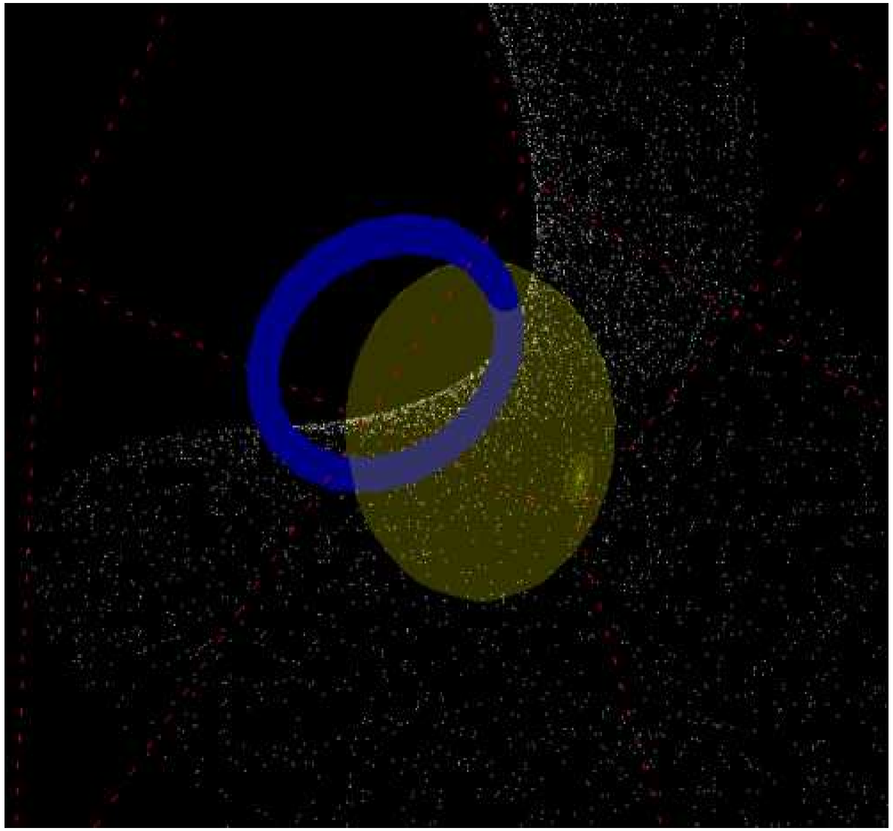


Curvature estimation for point clouds

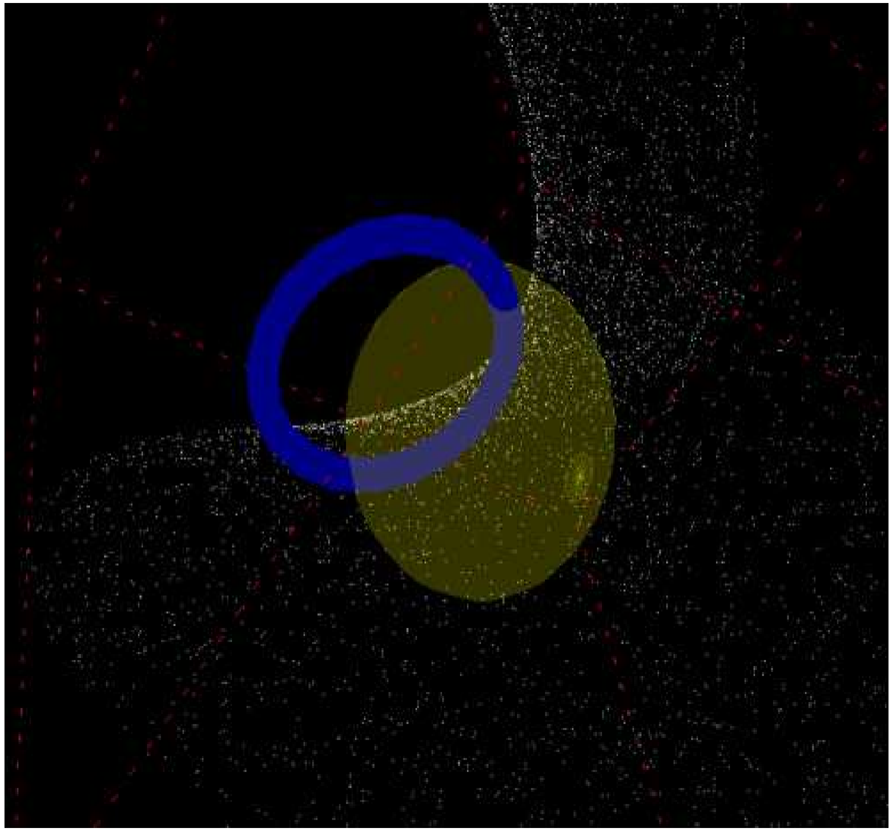
- Estimate osculating circle locally in point cloud
  - depending on different tangent directions

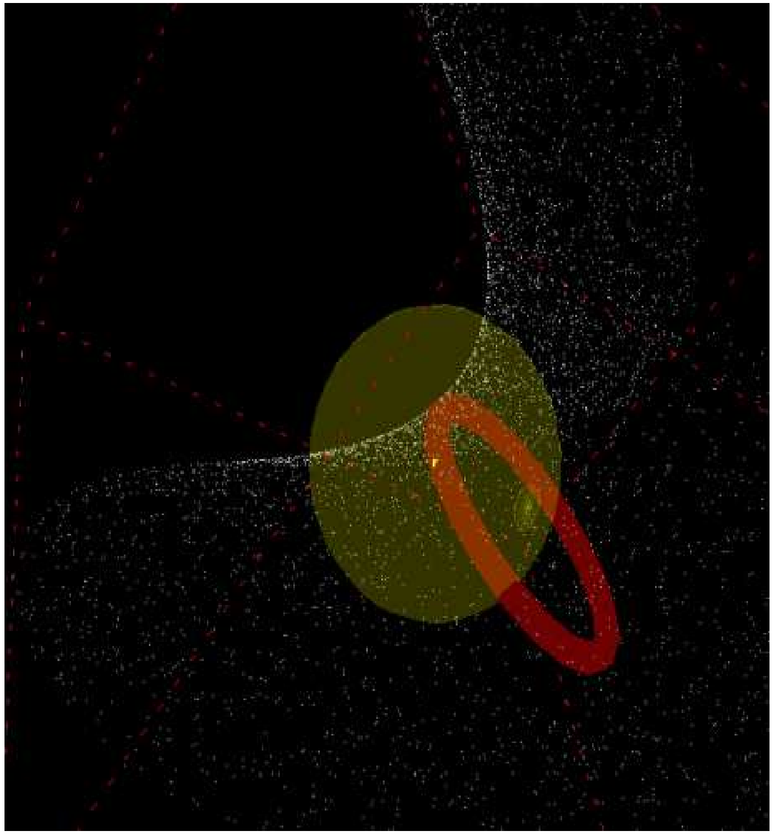
In more detail:

- Estimate osculating circle <u>or</u> line locally in point cloud depending on different tangent directions
  - Includes vanishing curvature



Example hyperbolic point

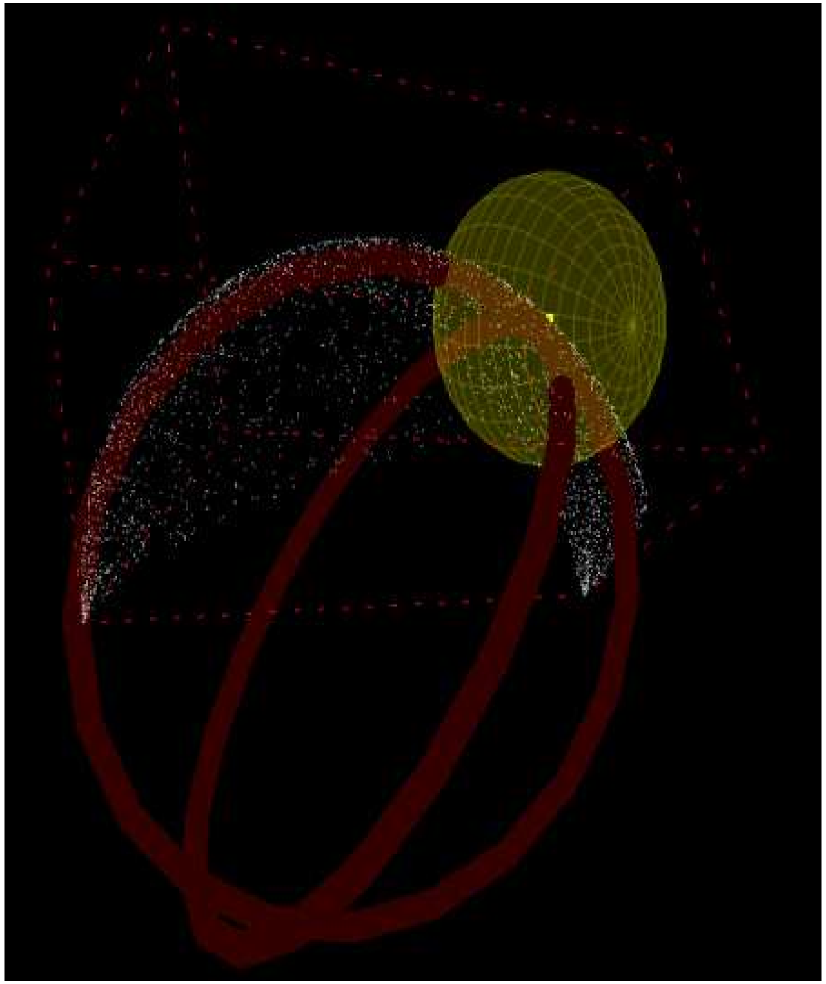




- Curvatures with different signs

Sphere example

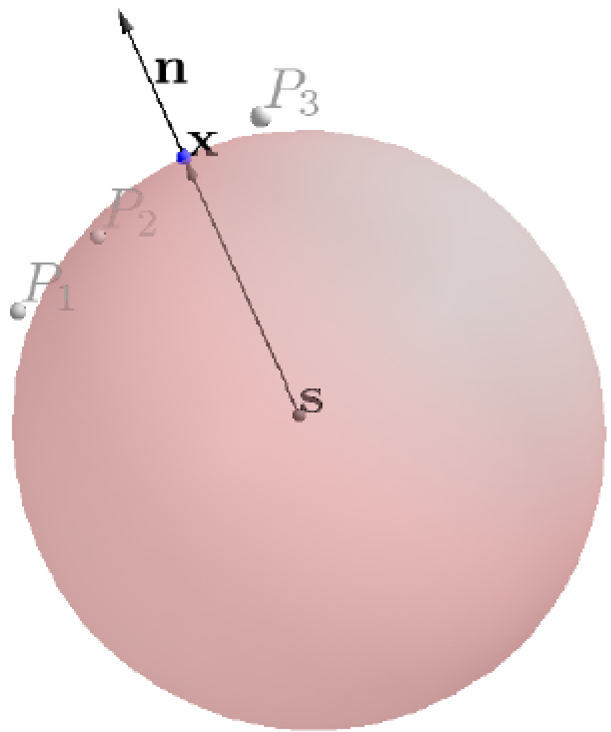
- Curvatures with same signs



Algorithm for circle/line fit

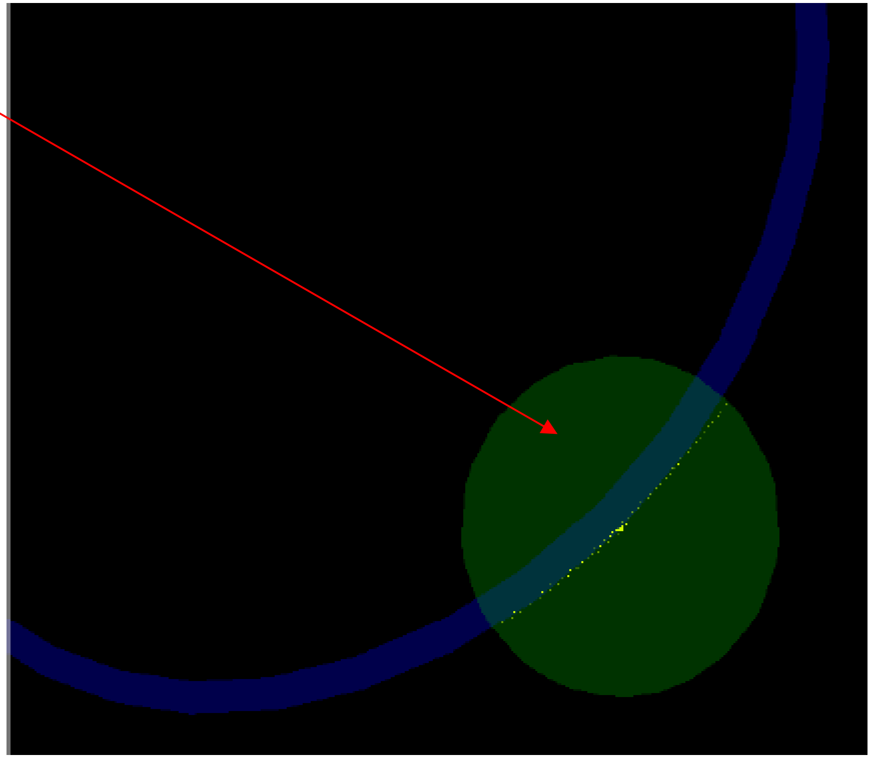
- Estimate normal vector **n** locally at **x**
- Determine points $\mathbf{P_i}$ in desired tangent direction
- Estimate <u>sphere</u> in the points $\mathbf{P_i}$
  - with center point in normal direction

-> the radius of the sphere describes the curvature in the desired direction



Curvature in one tangent direction

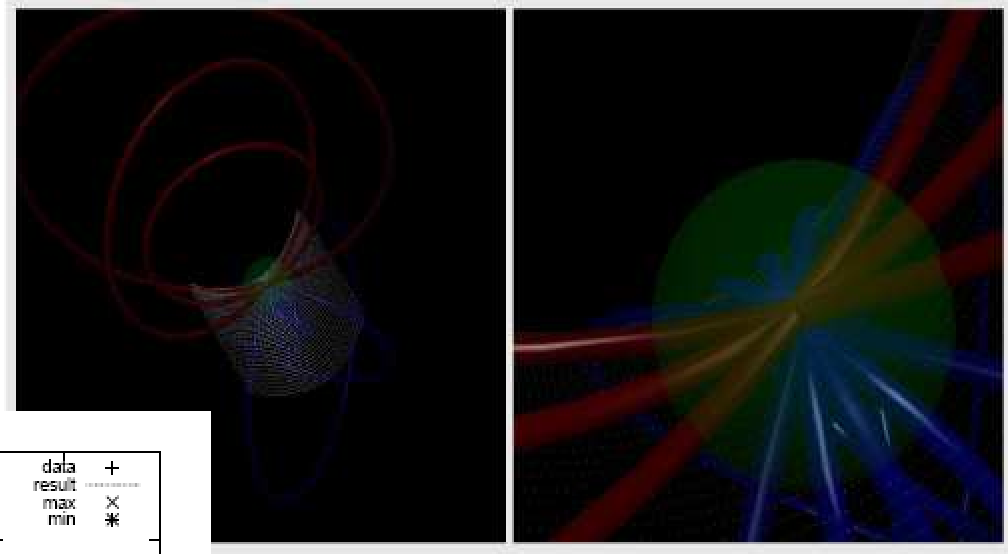
- In the estimation radius
  - Result is plane/line (no curvature)
  - Otherwise: osculating circle

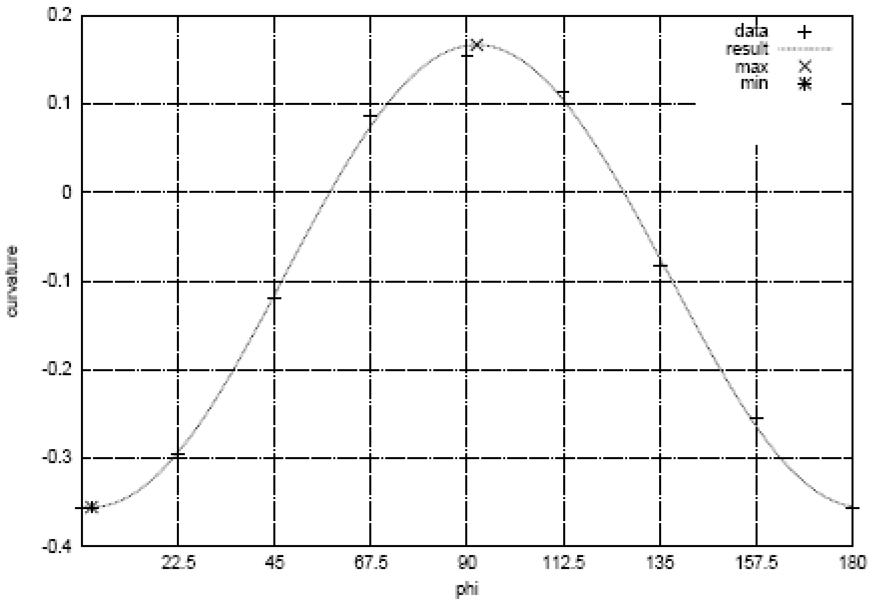


Curvature = circle fit in all directions

- Determine curvature in eight directions

- Approximate curvatures





Future work

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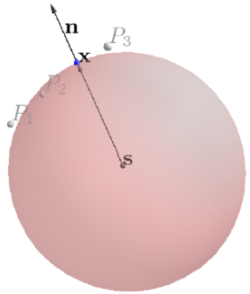
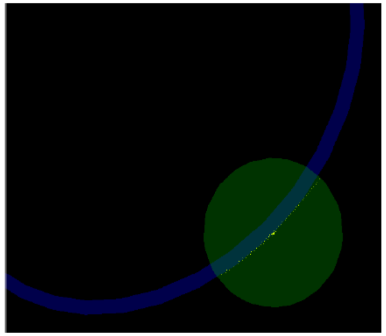
- Linear equation instead of Eigen vector determination
  - Bachelor thesis Roman Getto:
  - Verbesserte Hauptkrümmungs-bestimmung in Punktwolken durch optimiertes Schmiegekreisfitting auf Grundlage der Geometrischen Algebra





- Least Squares completely in GA

- Recognition of geometric objects (cylinder, torus etc.)

Thanks for your attention