Compass Ruler Algebra – and its Geometric Objects

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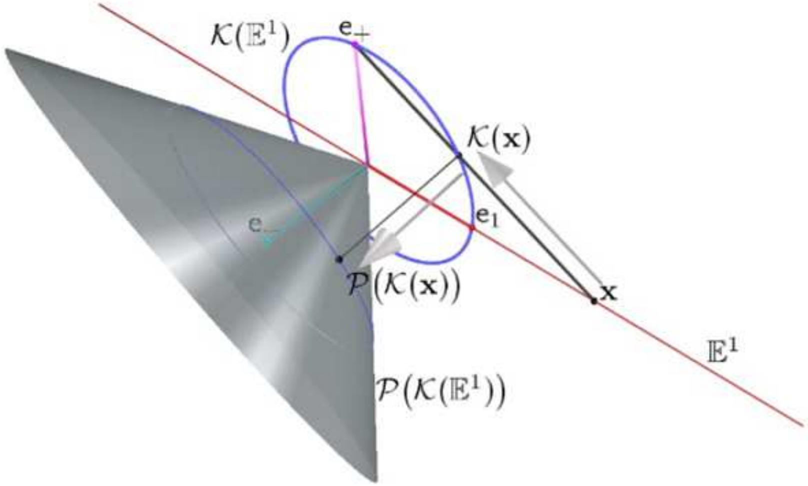

Compass Ruler Algebra

- The Algebraic Structure

The Compass Ruler Algebra $G_{3,1}$ uses the two Euclidean basis vectors e_1 and e_2 of the plane and two additional basis vectors e_+, e_- with positive and negative signatures, respectively, which means that they square to $+1$ as usual (e_+) and to -1 (e_-).

$$e_+^2 = 1, \qquad e_-^2 = -1, \qquad e_+ \cdot e_- = 0. \tag{5.1}$$

Another basis e_0, e_∞, with the following geometric meaning

e_0 represents the origin,

e_∞ represents infinity,

(see Sect. 5.9) can be defined with the relations

$$e_0 = \frac{1}{2}(e_- - e_+), \qquad e_\infty = e_- + e_+. \tag{5.2}$$

These new basis vectors are null vectors:

$$e_0^2 = e_\infty^2 = 0. \tag{5.3}$$

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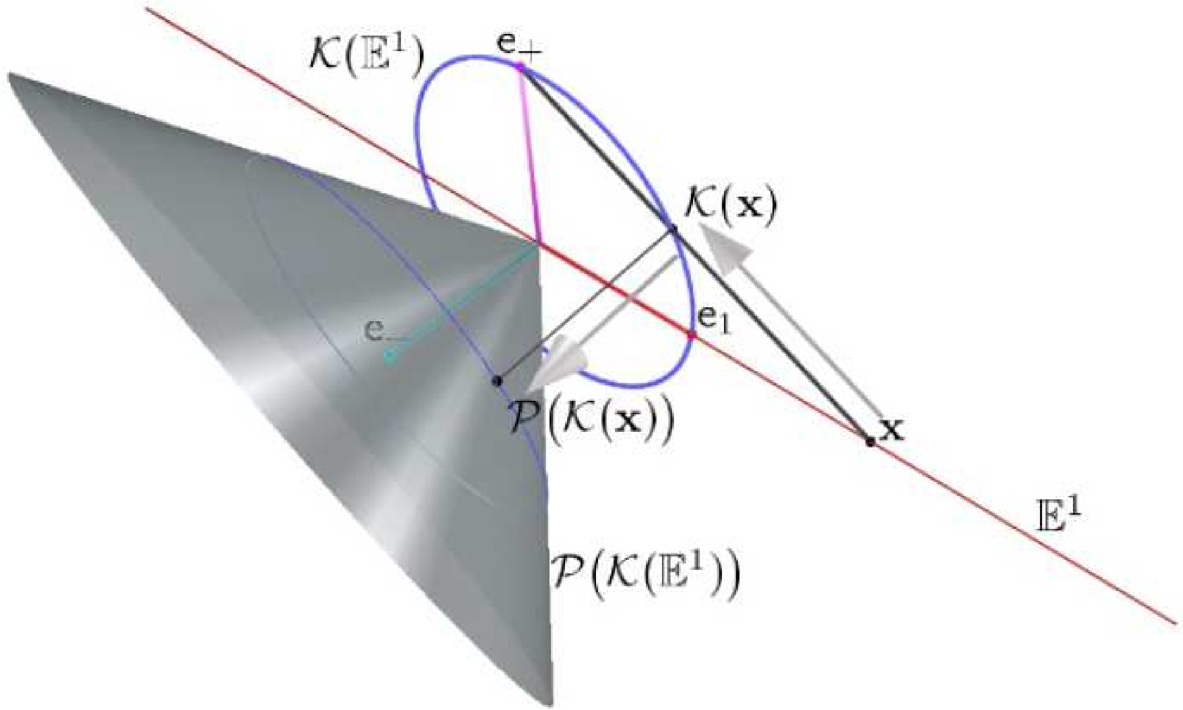

FIGURE 5.1 The mathematical model behind the Conformal Geometric Algebra of 1D space (image from [59]).

Compass Ruler Algebra

- The geometric objects

TABLE 5.1 The representations of the geometric entities of the Compass Ruler Algebra.

Entity	IPNS representation	OPNS representation
Point	$P = \mathbf{x} + \frac{1}{2}\mathbf{x}^2 e_\infty + e_0$	
Circle	$C = P - \frac{1}{2}r^2 e_\infty$	$C^* = P_1 \wedge P_2 \wedge P_3$
Line	$L = \mathbf{n} + d e_\infty$	$L^* = P_1 \wedge P_2 \wedge e_\infty$
Point pair	$P_p = C_1 \wedge C_2$	$P_p^* = P_1 \wedge P_2$

IPNS/OPNS

These entities have two algebraic representations: the (standard) **IPNS** (inner product null space) and the (dual) **OPNS** (outer product null space). The IPNS of the algebraic expression A are all the points X satisfying the equation

$$A \cdot X = 0. \tag{5.8}$$

The OPNS of the algebraic expression A are all the points X satisfying the equation

$$A \wedge X = 0. \tag{5.9}$$

These representations are duals of each other (a superscript asterisk denotes the dualization operator).

In the following, we present the representations of the basic geometric entities based on their null spaces.

Compass Ruler Algebra

- The Point

In order to represent points in Compass Ruler Algebra, the original 2D point

$$\mathbf{x} = x_1 e_1 + x_2 e_2 \tag{5.10}$$

is extended to a 4D vector by taking a linear combination of the 4D basis vectors e_1, e_2, e_∞, and e_0 according to the equation

$$P = \mathbf{x} + \frac{1}{2}\mathbf{x}^2 e_\infty + e_0. \tag{5.11}$$

where $\mathbf{x}^2$ is the inner product

$$\mathbf{x}^2 = (x_1 e_1 + x_2 e_2) \cdot (x_1 e_1 + x_2 e_2) = x_1^2 e_1^2 + 2x_1 x_2 \underbrace{(e_1 \cdot e_2)}_{0} + x_2^2 e_2^2 = x_1^2 + x_2^2. \tag{5.12}$$

For example, for the 2D origin $(x_1, x_2) = (0, 0)$ we get $P = e_0$.

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- The Point

In order to evaluate the geometric meaning of a point P with 2D coordinates (p_1, p_2), we compute its IPNS as its null space with respect to the inner product. The IPNS of P describes all the points X satisfying the equation

$$P \cdot X = 0. \tag{5.13}$$

The following GAALOPScript

Listing 5.1 *IPNSPoint.clu*: Computation of the IPNS of a point.

```
P = createPoint(p1,p2);
X = createPoint(x,y);
?IPPoint =  P.X;
```

computes this inner product and assigns it to the variable *IPPoint* (GAALOP computes all the variables indicated by a leading question mark). This resulting multivector is equal to

$$IPPoint_0 = \frac{1}{2}(-y^2 + 2 * p_2 * y - x^2 + 2 * p_1 * x - p_2^2 - p_1^2) \tag{5.14}$$

with the null space

$$y^2 - 2 * p_2 * y + x^2 - 2 * p_1 * x + p_2^2 + p_1^2 = 0$$

or

$$(y - p_2)^2 + (x - p_1)^2 = 0$$

describing exactly the point P.

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- The Line

A line is defined by

$$L = \mathbf{n} + de_\infty, \tag{5.15}$$

where $\mathbf{n} = n_1 e_1 + n_2 e_2$ refers to the 2D normal vector of the line L and d is the distance to the origin. The following GAALOPScript

Listing 5.2 *IPNSLine.clu*: Computation of the IPNS of a line.

```
X = createPoint(x,y);
L = n1*e1+n2*e2+d*einf;
?IP = X.L;
```

computes the inner product of a line L and a general point X. This results in the IPNS

$$n_1 * x + n_2 * y - d = 0 \tag{5.16}$$

which is a line with the corresponding normal vector (n_1, n_2) and distance d to the origin.

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- The Line

A line can also be defined with the help of two points that lie on it and the point at infinity:

$$L^* = P_1 \wedge P_2 \wedge e_\infty. \qquad (5.17)$$

Note that a line is a circle of infinite radius (see Sect 5.10).

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- The Circle

A circle can be represented with the help of its center point P and its radius r as

$$C = P - \frac{1}{2}r^2 e_\infty \quad (5.18)$$

or

$$C = \mathbf{x} + \frac{1}{2}\mathbf{x}^2 e_\infty + e_0 - \frac{1}{2}r^2 e_\infty \quad (5.19)$$

or

$$C = \mathbf{x} + \frac{1}{2}(\mathbf{x}^2 - r^2)e_\infty + e_0 \quad (5.20)$$

Note that the representation of a point is simply that of a circle of radius zero.

A circle can also be represented with the help of three points that lie on it, by

$$C^* = P_1 \wedge P_2 \wedge P_3. \quad (5.21)$$

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- The Circle

As an example, we compute the IPNS of the expression $e_0 - \frac{1}{2}r^2 e_\infty$, which means all the points X satisfying the following equation

$$\left(e_0 - \frac{1}{2}r^2 e_\infty\right) \cdot X = 0. \qquad (5.22)$$

The following GAALOPScript

Listing 5.3 *IPNSOriginCircle.clu*: Computation of the IPNS of an origin circle.

```
X = createPoint(x,y);
C = e0- 0.5*r*r*einf;
?Result = C.X;
```

computes this inner product. The resulting multivector is equal to the scalar value $\frac{1}{2}(x^2 + y^2 - r^2)$ with the null space

$$x^2 + y^2 - r^2 = 0, \qquad (5.23)$$

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- The radius of a circle

The following GAALOPScript

Listing 5.4 *CircleSquare.clu*: The square of a circle.

```
X = createPoint(x,y);
C = X - 0.5*r*r*einf;
?CSquare = C*C;
```

computes the square of a circle and results in

$$CSquare_0 = r * r,$$

which means the square of a circle equals to the square of its radius or

$$r = \sqrt{C^2}. \tag{5.24}$$

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- Normalized objects

Looking at the IPNS representations of point, circle and line of Table 5.1 we realize that they are all vectors of Compass Ruler Algebra. On the other hand, an arbitrary vector[4] must not have a representation as a geometric object. Considering an arbitrary vector

$$v = x_1 e_1 + x_2 e_2 + x_3 e_\infty + x_4 e_0 \tag{5.25}$$

and its null space

$$v \cdot X = 0 \tag{5.26}$$

we realize that

$$(cv) \cdot X = 0 \tag{5.27}$$

with an arbitrary scalar value $c \neq 0$ describing the same null space, since the IPNS equation $v \cdot X = 0$ is equivalent to the equation $(cv) \cdot X = c(v \cdot X) = 0$. This means that v and cv describe the same geometric object. Please notice that **this reasoning is not only true for vectors but also for arbitrary multivectors**, representing geometric objects.

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- Normalized objects

With this knowledge, we are able to determine what the geometric meaning of an arbitrary vector v is. If its e_0-component is zero, it represents a line

$$L = x_1 e_1 + x_2 e_2 + x_3 e_\infty. \tag{5.28}$$

Its normalized form can be computed by scaling with the length of the 2D vector (x_1, x_2), which can be expressed as

$$L_{normalized} = \frac{L}{|L|}. \tag{5.29}$$

This can be shown based on the following GAALOPScript

Listing 5.5 *normalizeLine.clu*: Normalization of a line.

```
line = n1*e1+n2*e2+n3*einf;
L = k*line;
?LAbs = abs(L);
```

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- Normalized objects

In the case of points and circles, the e_0-component equals to 1. This is why an arbitrary vector v has to be scaled by $x_4 \neq 0$.

$$C = \frac{x_1}{x_4}e_1 + \frac{x_2}{x_4}e_2 + \frac{x_3}{x_4}e_\infty + e_0. \tag{5.30}$$

This can be done based on the formula

$$C = -\frac{v}{v.e_\infty} \tag{5.31}$$

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Thanks a lot …