Compass Ruler Algebra – Angles/Distances Transformations

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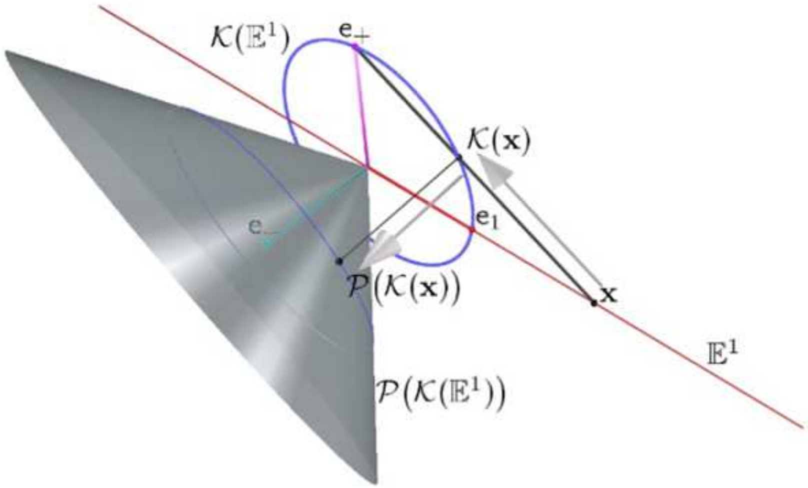

Compass Ruler Algebra

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5.9 THE MEANING OF E_0 AND E_∞

In order to evaluate the geometric meaning of e_0, we are able to compute its IPNS as its null space with respect to the inner product. The IPNS of e_0 describes all the points X satisfying the equation

$$e_0 \cdot X = 0. \tag{5.32}$$

The following GAALOPScript

Listing 5.7 *IPNSe0.clu*: Computation of the IPNS of e_0.

```
X = createPoint(x,y);
?Result = e0.X;
```

computes this inner product and assigns it to the variable *Result* (GAALOP computes all the variables indicated by a leading question mark). This resulting multivector is equal to the scalar value $x^2 + y^2$ with the null space

$$x^2 + y^2 = 0, \tag{5.33}$$

describing exactly the point at the origin.

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In order to evaluate the geometric meaning of e_∞, we assume an arbitrary Euclidean point $\mathbf{x} = x_1 e_1 + x_2 e_2$ (not equal to the origin) with a normalized Euclidean vector $\mathbf{n}$ in the direction of $\mathbf{x}$,

$$\mathbf{x} = t\mathbf{n},\ t > 0,\ \mathbf{n}^2 = 1 \tag{5.34}$$

with its representation P according to equation (5.11) and consider its limit $\lim_{t\to\infty}$. Another (homogeneous) representation of this point P is cP, its product with an arbitrary scalar value $c \neq 0$ (see Sect. 5.6).

Let us choose the arbitrary scalar value as $c = \frac{2}{\mathbf{x}^2}$ and consider $P' = \frac{2}{\mathbf{x}^2}P$,

$$P' = \frac{2}{\mathbf{x}^2}(\mathbf{x} + \frac{1}{2}\mathbf{x}^2 e_\infty + e_0), \tag{5.35}$$

$$P' = \frac{2}{\mathbf{x}^2}\mathbf{x} + e_\infty + \frac{2}{\mathbf{x}^2}e_0. \tag{5.36}$$

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$$P' = \frac{2}{\mathbf{x}^2}\mathbf{x} + e_\infty + \frac{2}{\mathbf{x}^2}e_0. \tag{5.36}$$

We use this form to compute the limit $\lim_{t\to\infty} P'$ for increasing $\mathbf{x}$. Since $\mathbf{x} = t\mathbf{n}$, we get

$$P' = \frac{2}{t^2\mathbf{n}^2}t\mathbf{n} + e_\infty + \frac{2}{t^2\mathbf{n}^2}e_0 \tag{5.37}$$

and, since $\mathbf{n}^2 = 1$,

$$P' = \frac{2}{t}\mathbf{n} + e_\infty + \frac{2}{t^2}e_0. \tag{5.38}$$

Based on this formula and the fact that P and P' represent the same Euclidean point, we can easily see that the point at infinity for any direction vector $\mathbf{n}$ is represented by e_∞:

$$\lim_{t\to\infty} P' = e_\infty. \tag{5.39}$$

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- Point pairs

Point pairs can be represented directly by the dual of the outer product of two points

$$Pp^* = P_1 \wedge P_2. \tag{5.49}$$

A point pair can also be defined by the intersection of two circles (one or both circles can also be lines, which are specific circles according to Sect. 5.10)

$$Pp = C_1 \wedge C_2 \tag{5.50}$$

TABLE 6.1 The two representations of point pairs.

IPNS representation	OPNS representation
$P_p = C_1 \wedge C_2$	$P_p^* = P_1 \wedge P_2$

We can use the following formula to extract the two points of the point pair Pp (see [8, 12]):

$$P_{1,2} = \frac{Pp^* \pm \sqrt{Pp^* \cdot Pp^*}}{e_\infty \cdot Pp^*} \tag{5.51}$$

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- The IPNS of the outer product of two vectors

First of all, we will show that the outer product of two vectors A and B really represents the intersection of the two objects represented by A an B. The IPNS of $A \wedge B$ is defined as the set of points X satisfying the following equation (see Sect. 5.6)

$$X \cdot (A \wedge B) = 0 \tag{6.1}$$

which is equivalent to

$$(X \cdot A)B - (X \cdot B)A = 0 \tag{6.2}$$

according to the rules of the inner product of a vector and a bivector (see Sect. 4.2.2). This can only be zero if

$$X \cdot A = 0 = X \cdot B \tag{6.3}$$

which means the IPNS of $A \wedge B$ equals all the points belonging to A and B.

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- The IPNS of the outer product of two vectors

What is the role of $e_1 \wedge e_2$ then? Since e_1 and e_2 represent two lines through the origin with normals e_1 and e_2, $e_1 \wedge e_2$ represents the intersection of these lines. Taking its dual

```
?PP = *(e1^e2);
```

with GAALOP results in

```
PP[10] = 1.0; // einf ^ e0
```

which means

$$(e_1 \wedge e_2)^* = e_\infty \wedge e_0, \tag{6.5}$$

which is the outer product of two specific points, namely the origin and infinity.

Compass Ruler Algebra

- The Intersection of two lines

Listing 6.1 *IntersectLines.clu*: Computation of the intersection of lines.

```
P = createPoint(p1,p2);
PPdual = P^einf;
?PP = *PPdual;

L = l1*e1+l2*e2+l3*einf;
M = m1*e1+m2*e2+m3*einf;

?I = L^M;
```

results in a point pair of the real intersecting point and the point at infinity

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The Intersection of two parallel lines

Listing 6.2 *IntersectParallelLines.clu*: Computation of the intersection of parallel lines.

```
n = n1*e1 + n2*e2;
L1 = n + d1*einf;
L2 = n + d2*einf;
?IL = L1^L2;
```

results in

$$\mathbf{v} \wedge e_\infty \tag{6.12}$$

(this is called a free vector in the literature [8]) with

$$\mathbf{v} = (d_2 - d_1)\mathbf{n}. \tag{6.13}$$

$(d_2 - d_1)$ describes the distance between the two lines with normal vector $\mathbf{n}$. This is why $\mathbf{v}$ describes the translation vector in order to translate one line into the other.

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- The Intersection of Circle-Line

Listing 6.3 *PointPairFromCircleandLine.clu*: computation of a point pair.

```
C = createPoint(c1,c2)-0.5*r*r*einf;
c = c1*e1+c2*e2;
n = n1*e1+n2*e2;
d = n.c;
L = n + d*einf;
?PP = 2*(C^L);
```

computes a point pair based on the intersection of a circle C with a line L going through the center point of the circle.

- This can be used for the interpretation of multivectors describing point pairs

Compass Ruler Algebra

- Distances and Angles

TABLE 7.1 Geometric meaning of the inner product of (normalized) lines, circles and points.

·	Line	Circle	Point
Line	Angle between lines Eq. (7.9)	Euclidean distance from center, Eq. (7.13)	Euclidean distance Eq. (7.6)
Circle	Euclidean distance from center, Eq. (7.13)	Distance measure Fig. 7.7	Distance measure Eq. (7.16)
Point	Euclidean distance Eq. (7.6)	Distance measure Eq. (7.16)	Distance Eq. (7.3)

Compass Ruler Algebra

- Distance between points

Listing 7.1 *DistancePointPoint.clu*: Computation of the inner product of two points.

```
P = createPoint(p1,p2);
Q = createPoint(q1,q2);
?Result = P.Q;
```

and results in

$$Result_0 = -0.5 * q2 * q2 + p2 * q2 - \frac{q1 * q1}{2} + p1 * q1 - \frac{p2 * p2}{2} - \frac{p1 * p1}{2}$$

or

$$P \cdot Q = -\frac{1}{2}(q_2^2 - 2p_2q_2 + q_1^2 + p_2^2 - 2p_1q_1 + p_1^2) \quad (7.1)$$

$$= -\frac{1}{2}((q_1 - p_1)^2 + (q_2 - p_2)^2) \quad (7.2)$$

$$= -\frac{1}{2}(\mathbf{q} - \mathbf{p})^2 \quad (7.3)$$

Compass Ruler Algebra

- Distance between point and line

```
P = createPoint(p1,p2);
L = n1*e1+n2*e2+d*einf;
?Result = P.L;
```

and results in

$$P \cdot L = p_1 n_1 + p_2 n_2 - d$$

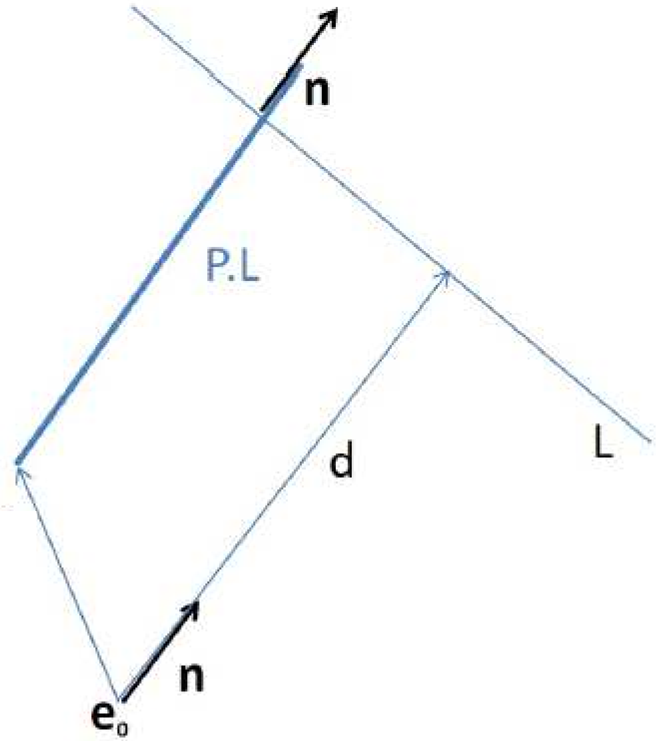
$$= \mathbf{p} \cdot \mathbf{n} - d, \qquad (7.6)$$

which represents the Euclidean distance between the point and the line, with a sign according to

$P \cdot L > 0$: $\mathbf{p}$ is on the normal $\mathbf{n}$ side of the line;

$P \cdot L = 0$: $\mathbf{p}$ is on the line;

$P \cdot L < 0$: $\mathbf{p}$ is on the opposite side of the normal $\mathbf{n}$.



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- Angles between lines

Listing 7.3 *AngleBetweenNormalizedLines.clu*: Computation of the inner product of two lines.

```
L1 = n11*e1+n12*e2+d1*einf;
L2 = n21*e1+n22*e2+d2*einf;
?Result = L1.L2;
?ResultDualLines = *L1.*L2;
```

resulting in

$$L_1 \cdot L_2 = \mathbf{n_1} \cdot \mathbf{n_2} \tag{7.7}$$

as well as for the dual lines

$$L_1^* \cdot L_2^* = \mathbf{n_1} \cdot \mathbf{n_2}. \tag{7.8}$$

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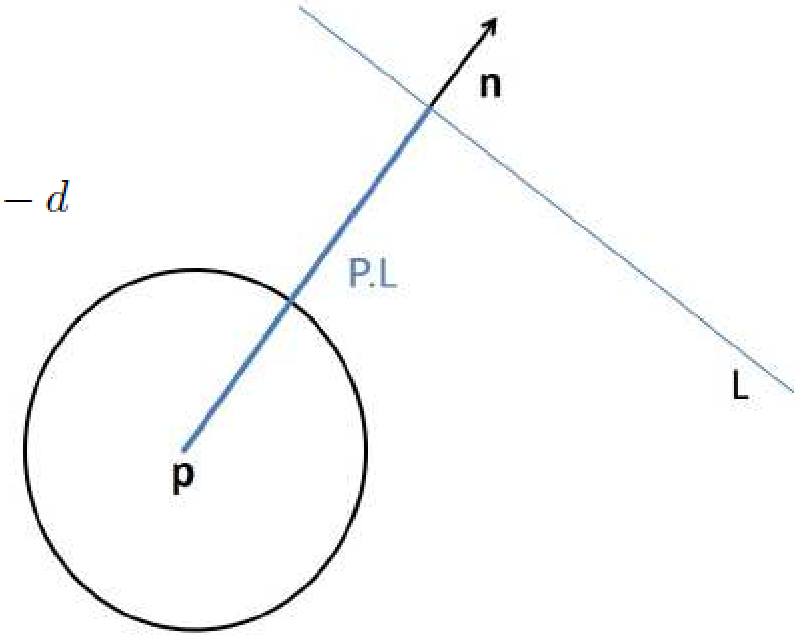
- Distance between a line and a circle

```
P = createPoint(p1,p2);
C = P - 0.5*r*r*einf;
L = n1*e1+n2*e2+d*einf;
?Result = L.C;
```

and results in

$$L \cdot C = n_1 p_1 + n_2 p_2 - d$$

$$= \mathbf{n} \cdot \mathbf{p} - d,$$



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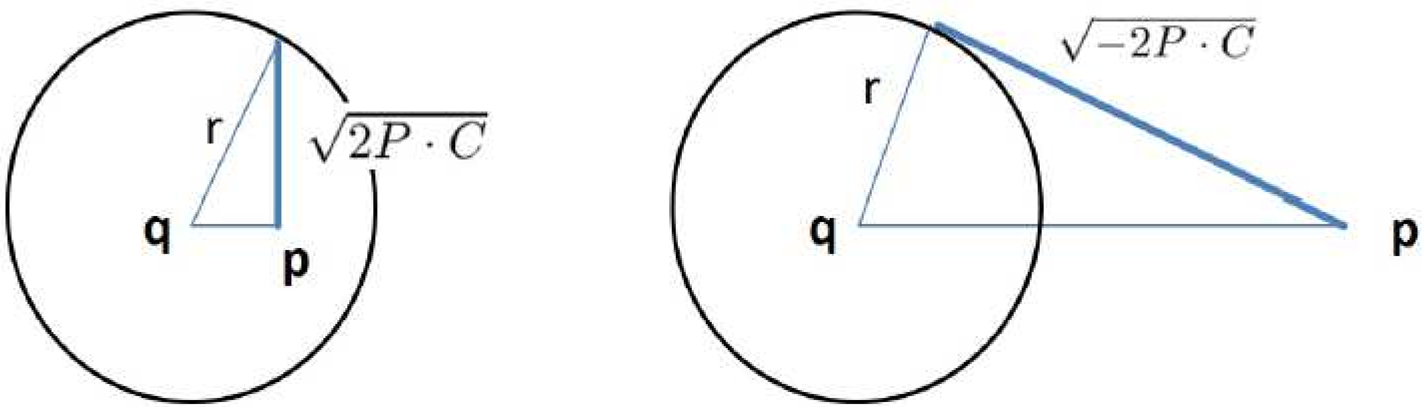
- Distance relations between a point and a circle



a) $r^2 = 2(P \cdot C) + (\mathbf{q} - \mathbf{p})^2$ b) $(\mathbf{q} - \mathbf{p})^2 = r^2 - 2(P \cdot C)$

FIGURE 7.3 The inner product of a point and a circle describes the distance of the bold segment depending on whether the point lies a) inside or b) outside the circle.

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- Distance relations between a point and a circle

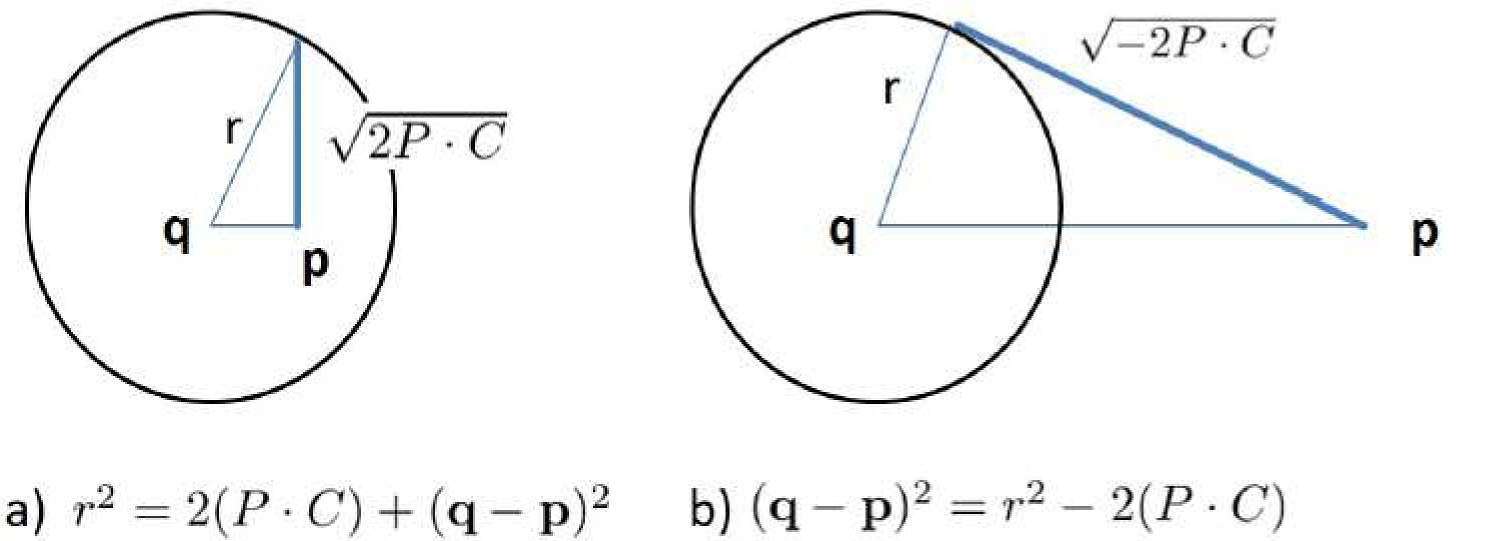

FIGURE 7.3 The inner product of a point and a circle describes the distance of the bold segment depending on whether the point lies a) inside or b) outside the circle.

Compass Ruler Algebra

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- Distance relations between a point and a circle

The following GAALOPScript computes the inner product of a point P and a circle C;

Listing 7.6 *DistancePointCircle.clu*: Computation of the inner product of a point and a circle.

```
P = createPoint(p1,p2);
Q = createPoint(q1,q2);
C=Q-0.5*r*r*einf;
Result = P.C;
```

and results in

$$P \cdot C = \frac{1}{2}r^2 - \frac{1}{2}(\mathbf{q} - \mathbf{p})^2 \tag{7.15}$$

or

$$2(P \cdot C) = r^2 - (\mathbf{q} - \mathbf{p})^2 \tag{7.16}$$

or

$$-2(P \cdot C) = (\mathbf{q} - \mathbf{p})^2 - r^2. \tag{7.17}$$

Fig. 7.3 illustrates this formula.

Compass Ruler Algebra

- Distance relations between two circles based on the inner product

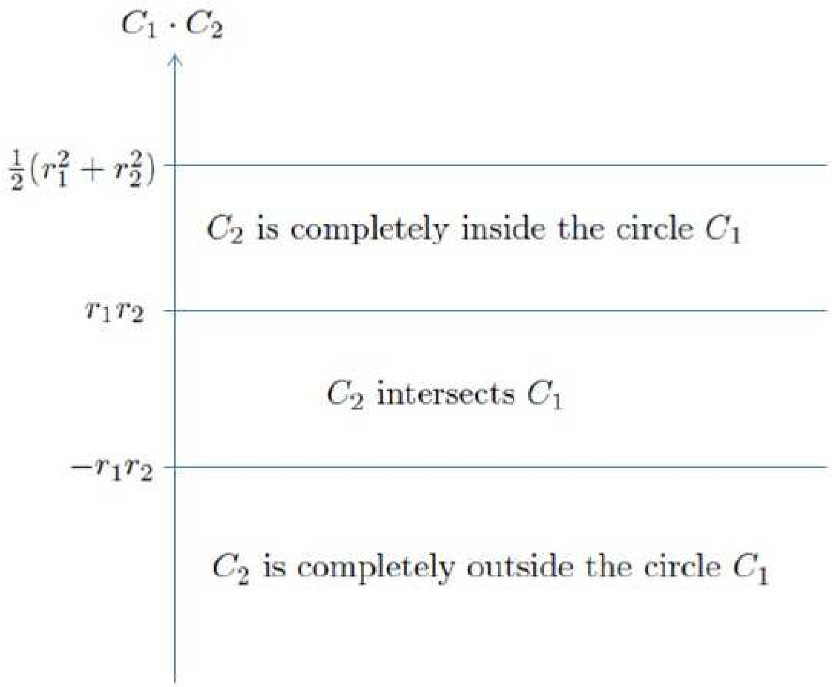

FIGURE 7.7 The geometric meaning of the inner product $C_1 \cdot C_2$ of two circles C_1 and C_2 depending on their radii r_1 and r_2.

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- Distance relations between two circles based on the inner product

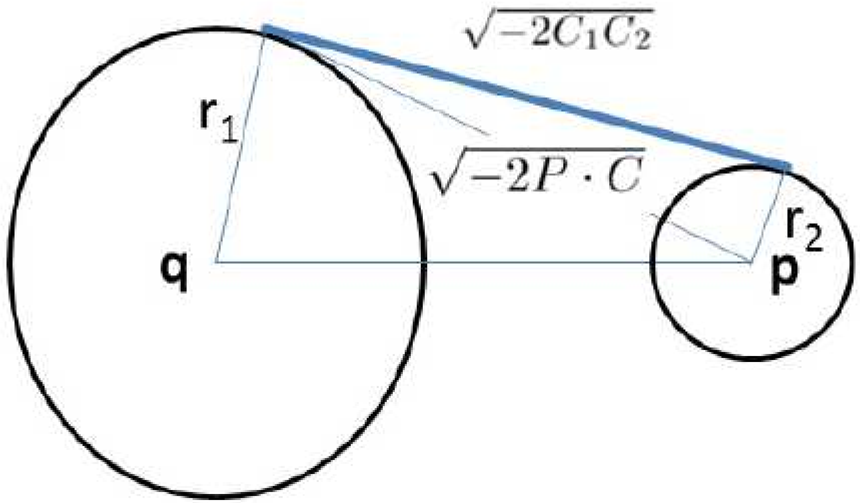


FIGURE 7.15 The bold segment describes the inner product of the two circles.

Compass Ruler Algebra

- Transformation of objects

TABLE 8.1 The description of transformations of a geometric object o in Compass Ruler Algebra.

Transformation	Operator	Usage
Reflection	Line $L = \mathbf{n} + de_\infty$	$o_L = -LoL$
Rotation	Rotor $R = \cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right) e_1 \wedge e_2$	$o_R = Ro\tilde{R}$
Translation	Translator $T = 1 - \frac{1}{2}\mathbf{t}e_\infty$	$o_T = To\tilde{T}$
Rigid Body Motion	Motor $M = \cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right) (P \wedge e_\infty)^*$	$o_M = Mo\tilde{M}$

Compass Ruler Algebra

- Transformation of objects

Listing 8.1 *ReflectCircleE1.clu*: Script for the reflection of the circle o at the line $L1 = e_1$ as well as at the line $L2 = e_2$.

```
o = createPoint(x,y)-0.5*r*r*einf;
L1 = e1;
L2 = e2;
?o1Refl = - L1 * o * L1;
?o2Refl = - L2 * o * L2;
```

$$o_{1Refl} = -xe_1 + ye_2 + \frac{1}{2}(x^2 + y^2 - r^2)e_\infty + e_0, \qquad (8.5)$$

which is the circle with a negated x-coordinate, meaning the circle is reflected at the y-axis,

and

$$o_{2Refl} = xe_1 - ye_2 + \frac{1}{2}(x^2 + y^2 - r^2)e_\infty + e_0, \qquad (8.6)$$

which is the circle with a negated y-coordinate, meaning the circle is reflected at the x-axis, as expected.

Compass Ruler Algebra

- Arbitrary reflections

Listing 8.3 *ArbitraryReflections.clu*: Script for the reflection of the circle o at an arbitrary line through the origin.

```
C = createPoint(x,y)-0.5*r*r*einf;
L1=n1*e1+n2*e2;
?o1Refl = -L1 * C * L1;
```

results in the reflection of a circle at an arbitrary line.

- Two consecutive reflections

Taking the two reflections together we can also compute the result for an arbitrary object o in one step as

$$o_{Refl2} = -L_2(-L_1 o L_1)L_2 = \underbrace{(L_2 L_1)}_{R} o \underbrace{(L_1 L_2)}_{\tilde{R}} \tag{8.8}$$

Compass Ruler Algebra

- Rotor based on Reflections

It is well known in mathematics that two consecutive reflections result in a rotation by twice the angle between the two lines of reflection. If we take two arbitrary normalized lines through the origin,

$$L_1 = n_1 e_1 + n_2 e_2 \tag{8.11}$$

$$L_2 = m_1 e_1 + m_2 e_2 \tag{8.12}$$

the reverse of the rotation operator can be computed according to Sect. 8.2 as

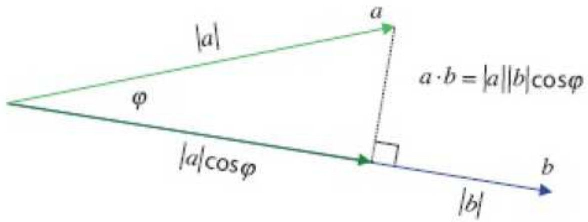
$$\tilde{R} = L_1 L_2 = (n_1 e_1 + n_2 e_2)(m_1 e_1 + m_2 e_2) \tag{8.13}$$

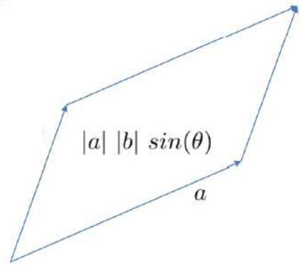
or

$$\tilde{R} = (n_1 e_1 + n_2 e_2) \cdot (m_1 e_1 + m_2 e_2) + (n_1 e_1 + n_2 e_2) \wedge (m_1 e_1 + m_2 e_2). \tag{8.14}$$

Compass Ruler Algebra

- Rotor based on Reflections





According to Sect. 4.2, $\tilde{R}$ can also be written

$$\tilde{R} = \cos(\theta) + e_1 \wedge e_2 \sin(\theta) \tag{8.15}$$

and the rotor R as its reverse

$$R = \cos(\theta) - e_1 \wedge e_2 \sin(\theta) \tag{8.16}$$

or with $\theta = \frac{\phi}{2}$

$$R = \cos\left(\frac{\phi}{2}\right) - e_1 \wedge e_2 \sin\left(\frac{\phi}{2}\right). \tag{8.17}$$

Based on this operator, the rotation of a geometric object o is performed with the help of the operation

$$o_{rotated} = Ro\tilde{R}. \tag{8.18}$$

Compass Ruler Algebra

We will show as follows that the operator R of Eq. 8.17 is equivalent to the operator

$$R = e^{-\frac{\phi}{2} e_1 \wedge e_2} \qquad (8.19)$$

also describing a **rotor** for a rotation around the origin with the rotation angle ϕ.

With the help of a Taylor series, we can write

$$R = 1 + \frac{-e_1 \wedge e_2 \frac{\phi}{2}}{1!} + \frac{(-e_1 \wedge e_2 \frac{\phi}{2})^2}{2!} + \frac{(-e_1 \wedge e_2 \frac{\phi}{2})^3}{3!} + \frac{(-e_1 \wedge e_2 \frac{\phi}{2})^4}{4!} + \ldots$$

or

$$R = 1 - \frac{e_1 \wedge e_2 \frac{\phi}{2}}{1!} + \frac{(e_1 \wedge e_2 \frac{\phi}{2})^2}{2!} - \frac{(e_1 \wedge e_2 \frac{\phi}{2})^3}{3!} + \frac{(e_1 \wedge e_2 \frac{\phi}{2})^4}{4!} + \ldots$$

or, according to $i^2 = (e_1 \wedge e_2)^2 = e_1 e_2 \underbrace{e_1 e_2}_{-e_2 e_1} = -e_1 \underbrace{e_2 e_2}_{1} e_1 = -\underbrace{e_1 e_1}_{1} = -1$,

$$R = 1 - \frac{(\frac{\phi}{2})^2}{2!} + \frac{(\frac{\phi}{2})^4}{4!} - \frac{(\frac{\phi}{2})^6}{6!} + \ldots - e_1 \wedge e_2 \frac{\frac{\phi}{2}}{1!} + e_1 \wedge e_2 \frac{(\frac{\phi}{2})^3}{3!} - e_1 \wedge e_2 \frac{(\frac{\phi}{2})^5}{5!} + \ldots,$$

and therefore

$$R = \cos\left(\frac{\phi}{2}\right) - e_1 \wedge e_2 \sin\left(\frac{\phi}{2}\right). \qquad (8.20)$$

Compass Ruler Algebra

- The Translator

In Compass Ruler Algebra, a translation can be expressed in a multiplicative way with the help of a **translator** T defined by

$$T = e^{-\frac{1}{2}\mathbf{t}e_\infty}, \tag{8.21}$$

where $\mathbf{t}$ is a vector

$$\mathbf{t} = t_1 e_1 + t_2 e_2. \tag{8.22}$$

Application of the Taylor series

$$T = e^{-\frac{1}{2}\mathbf{t}e_\infty} = 1 + \frac{-\frac{1}{2}\mathbf{t}e_\infty}{1!} + \frac{(-\frac{1}{2}\mathbf{t}e_\infty)^2}{2!} + \frac{(-\frac{1}{2}\mathbf{t}e_\infty)^3}{3!} + \ldots \tag{8.23}$$

and the property $(e_\infty)^2 = 0$ results in the translator

$$T = 1 - \frac{1}{2}\mathbf{t}e_\infty. \tag{8.24}$$

Compass Ruler Algebra

- The general rotation

In Compass Ruler Algebra, a rigid body motion is a general rotation, including both a rotation and a translation as described by

$$M = TR\tilde{T}, \tag{8.25}$$

where R is a rotor, T is a translator and M is the resulting motor. A rigid body motion of an object o is described by

$$o_M = Mo\tilde{M}. \tag{8.26}$$

The following GAALOPScript

Listing 8.4 *computeMotor.clu*: Computation of a general rotation.

```
R = r1 - r2* (e1^e2);
T = 1-0.5*(t1*e1+t2*e2)*einf;
?M = T*R* ~T;
```

results in

$$M_0 = r1$$
$$M_5 = -r2$$
$$M_6 = -r2 * t2$$
$$M_8 = r2 * t1$$

Compass Ruler Algebra

- The general rotation M

$$M = r_1 - r_2 e_1 \wedge e_2 - r_2 t_2 e_1 e_\infty + r_2 t_1 e_2 e_\infty \tag{8.27}$$

or

$$M = r_1 - r_2(e_1 \wedge e_2 - t_2 e_1 e_\infty + t_1 e_2 e_\infty) \tag{8.28}$$

or according to Eq. (6.6)

$$M = r_1 - r_2(P \wedge e_\infty)^* \tag{8.29}$$

with P being the conformal point of the 2D-Point (t_1, t_2).

Since r_1 and r_2 are the parameters of a rotations, this can be written in the form

$$M = \cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right)(P \wedge e_\infty)^* \tag{8.30}$$

or

$$M = \cos\left(\frac{\phi}{2}\right) - \sin\left(\frac{\phi}{2}\right) L \tag{8.31}$$

with L as the **point of rotation**

$$L = (P \wedge e_\infty)^* \tag{8.32}$$

Compass Ruler Algebra

- Inversion and the center of a circle

Inversions are reflections not at lines but at circles. We saw in Sect. 3.4.1.3 that geometric objects resulting from inversions of lines and circles at a circle C are circles. If the objects to be inverted at a circle C move away towards infinity, the resulting circle seems to converge to the center point of C, which means it seems that the center of a circle can be computed based on the sandwich product

$$P = Ce_{\infty}C. \tag{8.35}$$

describing the inversion of infinity at the circle C. We can show that with the following GAALOPScript

Compass Ruler Algebra

- Inversion and the center of a circle

Listing 8.8 *CircleCenterProof.clu*: Computation of the center point of a circle.

```
P = createPoint(p1,p2);
Circle = P-0.5*r*r*einf;
?PC = Circle*einf*Circle;
```

resulting in the point

$$P_C = -2\left(\mathbf{p} + \frac{1}{2}\mathbf{p}^2 e_\infty + e_0\right) \tag{8.36}$$

with a homogeneous scaling factor of -2.

This sandwich product can also be used to obtain the centers of point pairs. Note, that point pairs are specific lower-dimensional circles in Compass Ruler Algebra. The center of a point pair can be computed from

$$P = P_p e_\infty P_p. \tag{8.37}$$

Inversion Line-Circle

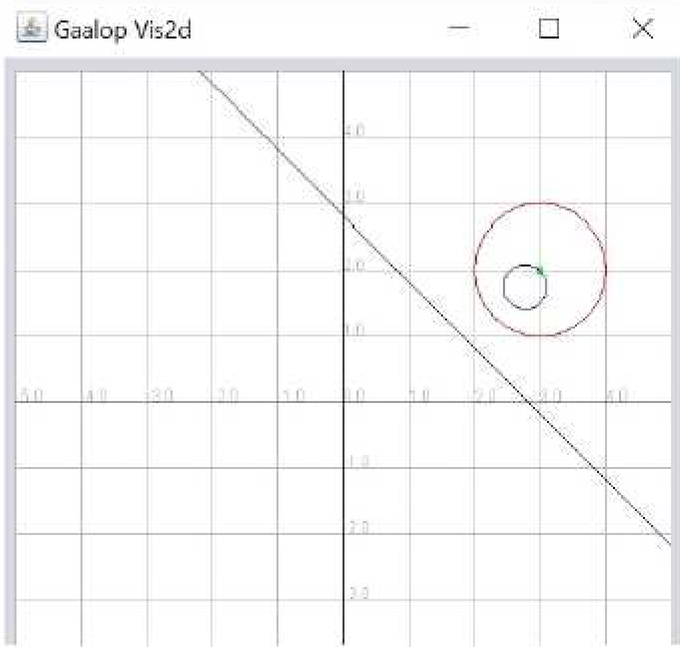


FIGURE 3.19 Visualization of LineInversion.clu (with t =2): the inversion of a line at a circle results in a circle.

Inversion Line-Circle

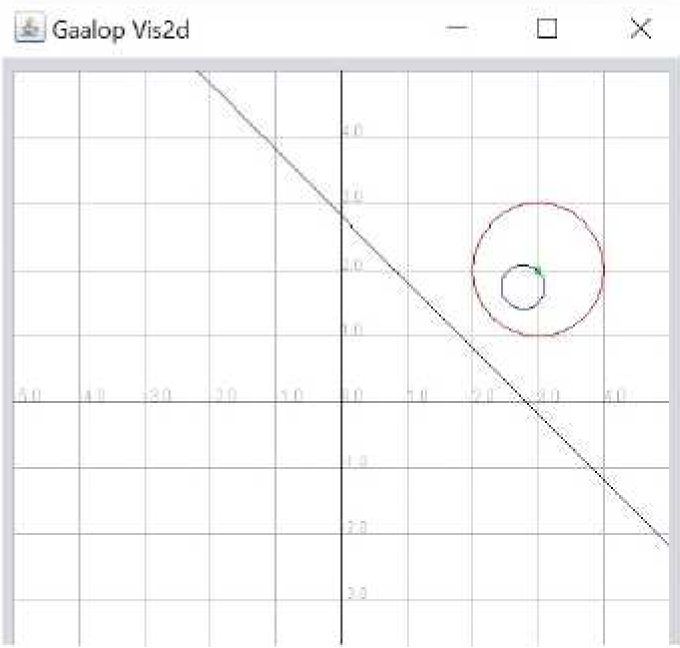


FIGURE 3.19 Visualization of LineInversion.clu (with t =2): the inversion of a line at a circle results in a circle.

- … through the origin of the circle (and vice-versa)

Inversion Line-Circle

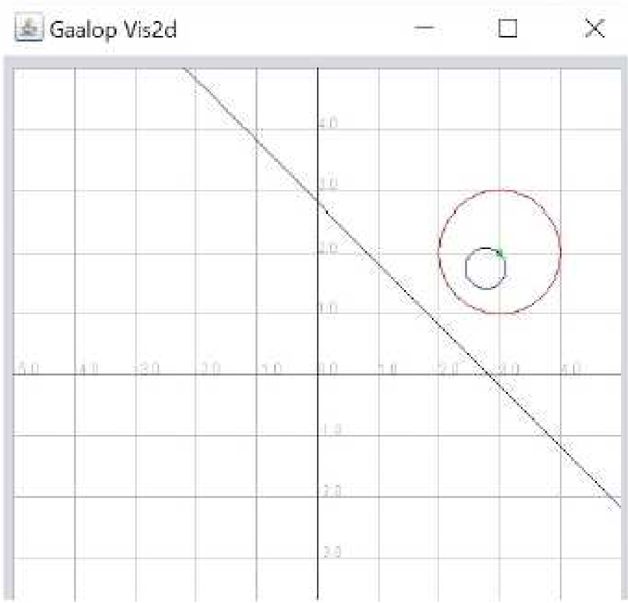


FIGURE 3.19 Visualization of LineInversion.clu (with t =2): the inversion of a line at a circle results in a circle.

- … through the origin of the circle (and vice-versa)

How to proof the vice-versa?

Thanks a lot …