Geometric Algebra – Mathematical Basics

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Termine

Termin	Themen
17.04.19	Einführung
24.04.19	Tutorial
08.05.19	GAALOP / Arbitrary Algebras
15.05.19	Mathematical Basics
22.05.19	fällt aus (Workshop Brasilien)
29.05.19	CGA
05.06.19	CGA
19.06.19	Fällt aus (Computer Graphics International)
26.06.19	Fällt aus (Computer Graphics International)
03.07.19	
10.07.19	
17.07.19	

2D Euclidean Geometric Algebra

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TABLE 4.1 The four basis blades of 2D Euclidean Geometric Algebra. This algebra consists of basic algebraic objects of grade (dimension) 0, the scalar, of grade 1 (the two basis vectors e_1 and e_2) and of grade 2 (the bivector $e_1 \wedge e_2$), which can be identified with the imaginary unit i squaring to -1.

Blade	Grade
1	0
e_1	1
e_2	1
$e_1 \wedge e_2$	2

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- The products of Geometric Algebra

TABLE 4.2 Notations for the Geometric Algebra products

Notation	Meaning
AB	Geometric product of A and B
$A \wedge B$	Outer product of A and B
$A \cdot B$	Inner product of A and B

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- The Outer Product

TABLE 4.3 Properties of the outer product $\wedge$ of vectors

Property	Meaning
Anti-Commutativity	$a \wedge b = -(b \wedge a)$
Distributivity	$a \wedge (b + c) = a \wedge b + a \wedge c$
Associativity	$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

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- The Outer Product

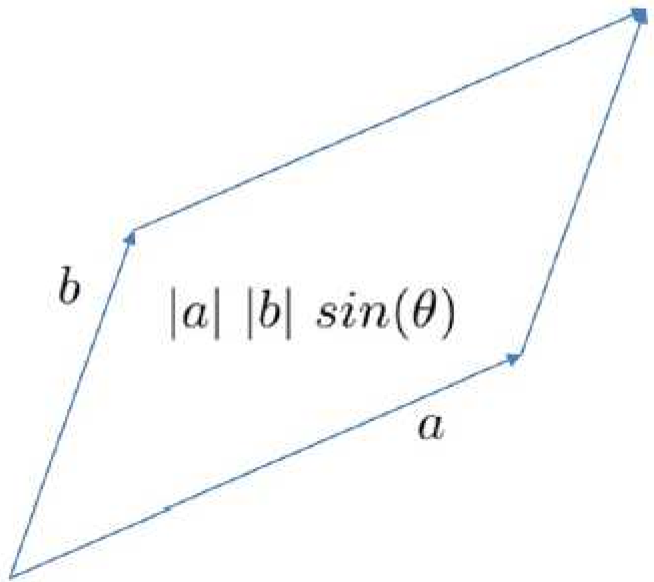

FIGURE 4.1 Magnitude of blade $a \wedge b$ is the area of the parallelogram spanned by a and b [53].

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- The Outer Product

Computation example

We compute the outer product of two vectors:

$$c = (e_1 + e_2) \wedge (e_1 - e_2) \tag{4.4}$$

can be transformed based on distributivity to

$$c = (e_1 \wedge e_1) - (e_1 \wedge e_2) + (e_2 \wedge e_1) - (e_2 \wedge e_2); \tag{4.5}$$

since $u \wedge u = 0$,

$$c = -(e_1 \wedge e_2) + (e_2 \wedge e_1), \tag{4.6}$$

and because of anti-commutativity,

$$c = -(e_1 \wedge e_2) - (e_1 \wedge e_2) \tag{4.7}$$

or

$$c = -2(e_1 \wedge e_2). \tag{4.8}$$

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- The Inner Product

While the outer product is anti-commutative, the inner product is commutative. For Euclidean spaces, the inner product of two vectors is the same as the well-known Euclidean scalar product of two vectors.

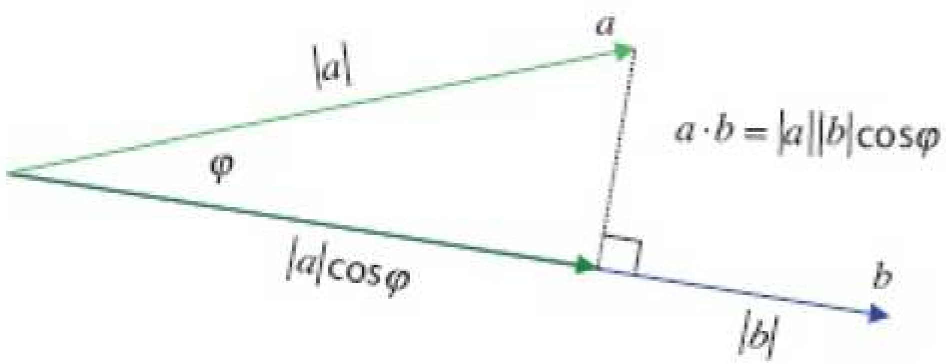

FIGURE 4.2 Scalar product of two vectors a and b.

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- The Geometric Product

The geometric product is an amazingly powerful operation, which is used mainly for the handling of transformations. The geometric product of vectors is a combination of the outer product and the inner product. The geometric product of u and v is denoted by uv (please notice that for the geometric product no specific symbol is used). For vectors u and v, the geometric product uv can be defined as the sum of outer and inner product

$$uv = u \wedge v + u \cdot v. \tag{4.13}$$

We derive the following for the inner and outer products:

$$u \cdot v = \frac{1}{2}(uv + vu), \tag{4.14}$$

$$u \wedge v = \frac{1}{2}(uv - vu). \tag{4.15}$$

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- The Geometric Product

Computation example: What is the square of a vector?

$$u^2 = uu = \underbrace{u \wedge u}_{0} + u \cdot u = u \cdot u \qquad (4.16)$$

for example

$$e_1^2 = e_1 \cdot e_1 = 1. \qquad (4.17)$$

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- The Imaginary Unit

TABLE 4.4 Multiplication table of 2D Euclidean Geometric Algebra.

	1	e_1	e_2	$e_1 \wedge e_2$
1	1	e_1	e_2	$e_1 \wedge e_2$
e_1	e_1	1	$e_1 \wedge e_2$	e_2
e_2	e_2	$-e_1 \wedge e_2$	1	$-e_1$
$e_1 \wedge e_2$	$e_1 \wedge e_2$	$-e_2$	e_1	-1

Since $e_1 e_2 = e_1 \wedge e_2 + \underbrace{e_1 \cdot e_2}_{0} = e_1 \wedge e_2$,

$$i^2 = (e_1 \wedge e_2)^2 = (e_1 e_2)\underbrace{(e_1 e_2)}_{-e_2 e_1} = -e_1 \underbrace{e_2 e_2}_{1} e_1 = -\underbrace{e_1 e_1}_{1} = -1 \qquad (4.18)$$

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- The Inverse

The **inverse** of a blade A is defined by

$$AA^{-1} = 1.$$

The inverse of a vector v, for instance, is

$$v^{-1} = \frac{v}{v \cdot v}.$$

Proof:

$$v\frac{v}{v \cdot v} = \frac{v \cdot v}{v \cdot v} = 1.$$

Example 1 The inverse of the vector $v = 2e_1$ results in $0.5e_1$, since $v \cdot v = 2$.

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- The Inverse

Example 2 The inverse of the (Euclidean) pseudoscalar $1/I$ is the negative of the pseudoscalar $(-I)$.

Proof:

$$II = (e_1 \wedge e_2)(e_1 \wedge e_2) = -1$$

$$\rightarrow II(I^{-1}) = -I^{-1}$$

$$\rightarrow I(II^{-1}) = -I^{-1}$$

$$\rightarrow I = -I^{-1}$$

$$\rightarrow I^{-1} = -I.$$

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- The Dual

Since the geometric product is **invertible**, divisions by algebraic expressions are possible.
The **dual** of an algebraic expression is calculated by dividing it by the pseudoscalar I. In the following, the dual of the pseudoscalar $e_1 \wedge e_2$ is calculated. A superscript $*$ means the dual operator.

$$(e_1 \wedge e_2)^* = (e_1 \wedge e_2)(e_1 \wedge e_2)^{-1}$$

$$(e_1 \wedge e_2)^* = (e_1 \wedge e_2)\underbrace{(e_1 \wedge e_2)^{-1}}_{-(e_1 \wedge e_2)}$$

$$(e_1 \wedge e_2)^* = -(e_1 \wedge e_2)(e_1 \wedge e_2)$$

$$(e_1 \wedge e_2)^* = -\underbrace{(e_1 \wedge e_2)(e_1 \wedge e_2)}_{-1}$$

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- The Reverse

The reverse of a multivector is the multivector with reversed order of the outer product components; for instance the reverse of $1 + e_1 \wedge e_2$ is $1 + e_2 \wedge e_1$ or $1 - e_1 \wedge e_2$.

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Thanks a lot …